Number systems. Binary arithmetic

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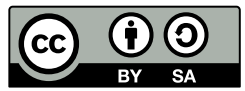
Institute of Informatics



Binary arithmetic

Representable digits in our hardware: **0** and **1**

System:	dec	bin	oct	hex
Base:	10	2	8	16

Binary arithmetic

Representable digits in our hardware: **0** and **1**

System: / Base:	dec 10	bin 2	oct 8	hex 16
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	4	**100**		
	5			
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Expressing numbers

In base b, an m-digit number is represented as a polynomial:

$$N = a_{m-1}b^{m-1} + a_{m-2}b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

Example:

$$\begin{aligned} 110101_2 &= 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 32_{10} + 16_{10} + 4_{10} + 1_{10} \\ &= 53_{10} \end{aligned}$$

Another example:

$$16_{16} = 1 \cdot 16^1 + 6 \cdot 16^0 = 16_{10} + 6_{10} = 22_{10}$$

Expressing fractions

In base b, a fraction $f = 0.a_1 a_2 \ldots a_m$ is represented as a polynomial:

$$f = a_1 b^{-1} + a_2 b^{-2} + \cdots + a_{m-1} b^{-m+1} + a_m b^{-m}$$

Example:

$$\begin{aligned} 0.0110101_2 &= 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} + 0 \cdot 2^{-4} + \\ &\quad + 1 \cdot 2^{-5} + 0 \cdot 2^{-6} + 1 \cdot 2^{-7} \\ &= \frac{1}{4_{10}} + \frac{1}{8_{10}} + \frac{1}{32_{10}} + \frac{1}{128_{10}} \\ &= 0.4140625_{10} \end{aligned}$$

Converting into other number systems

Division by the system's base gives reminder that is the last digit of the number:

$$\begin{aligned} N &= a_{m-1}b^{m-1} + a_{m-2}b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\ &= b(a_{m-1}b^{m-2} + a_{m-2}b^{m-3} + \cdots + a_2 b + a_1) + a_0 \end{aligned}$$

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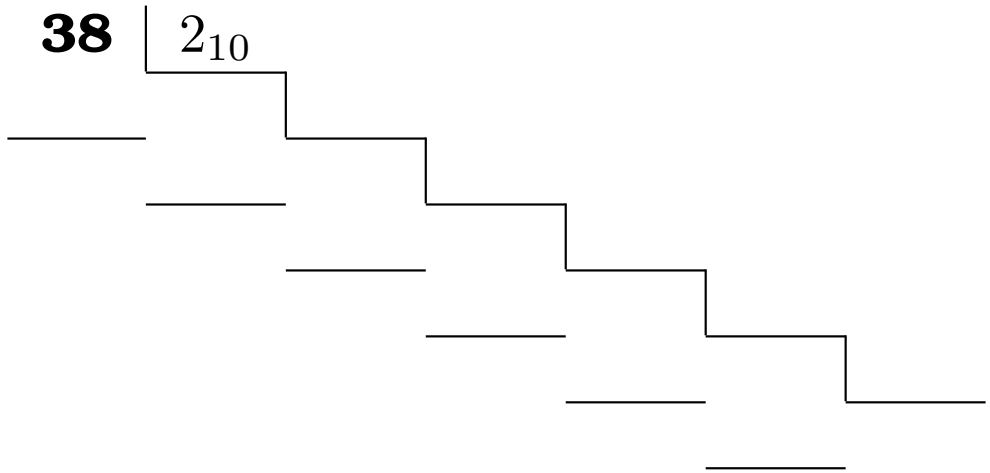
Division by the system's base gives reminder that is the last digit of the number:

$$100110_2/10_2 = 10011.0_2 \quad (\text{quotient} = 10011_2, \text{reminder} = 0)$$

$$10011_2/10_2 = 1001.1_2 \quad (\text{quotient} = 1001_2, \text{reminder} = 1)$$

Thus, repeated division by a *base* gives us digits in that base as *reminders*:



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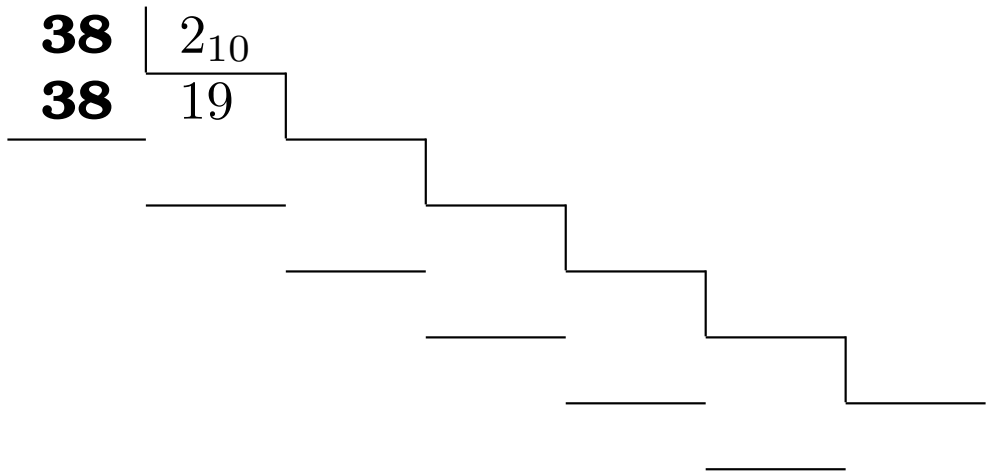

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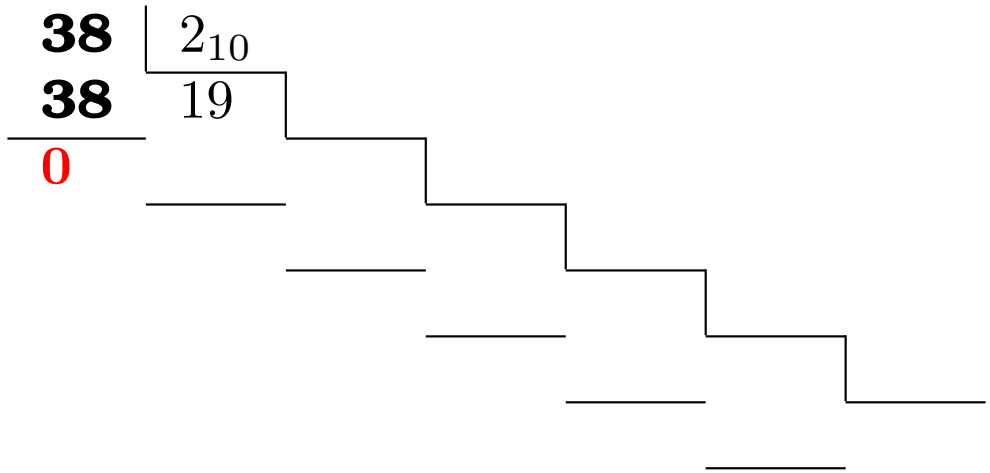

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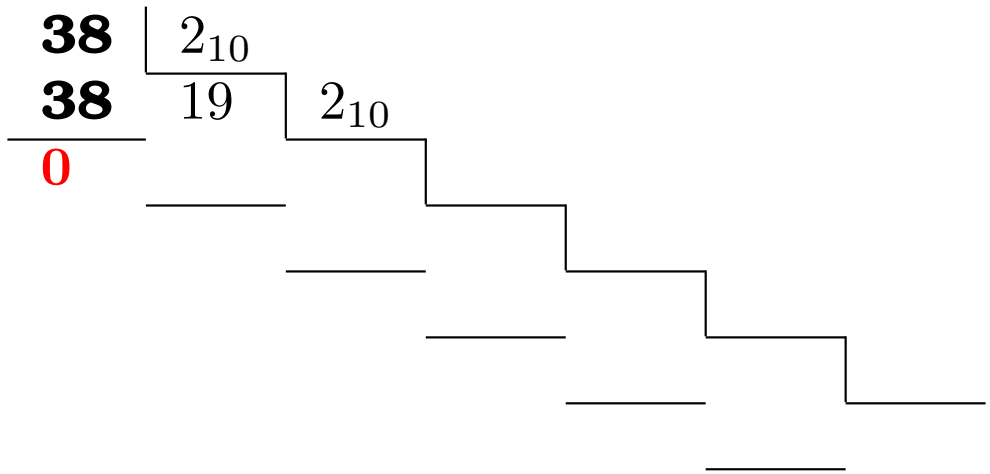

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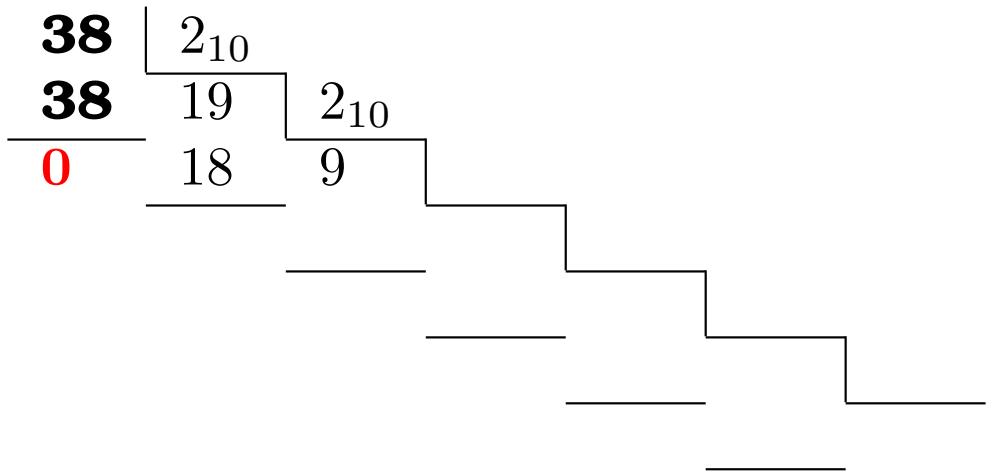

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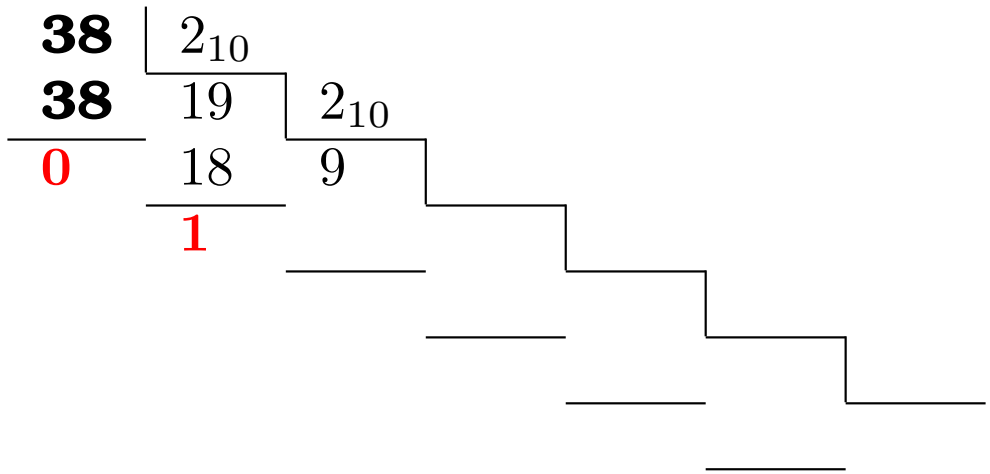

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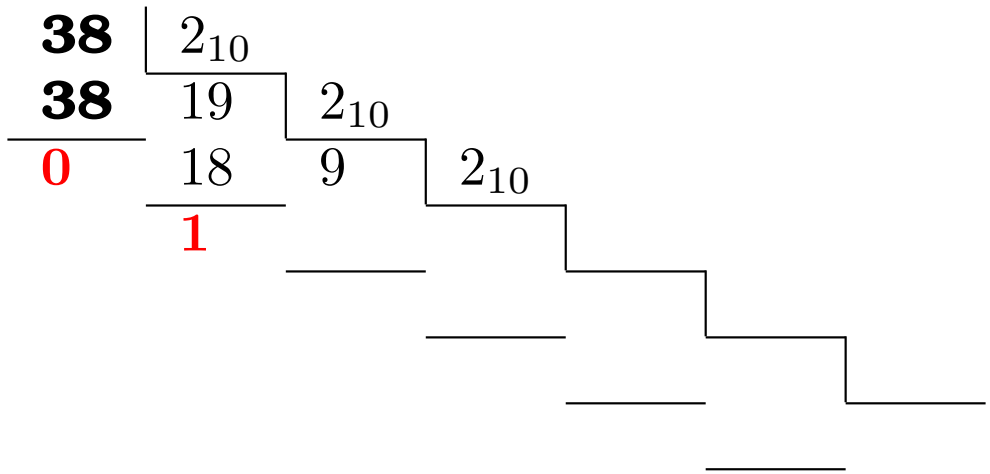

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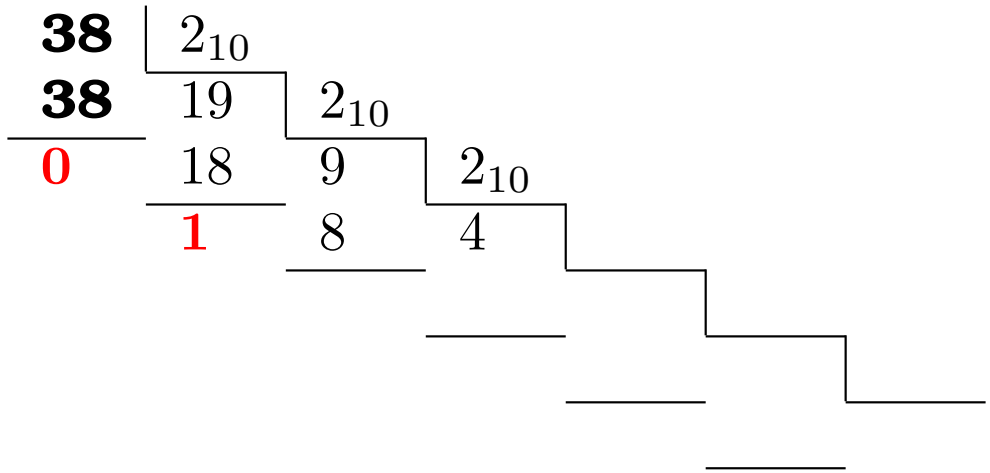

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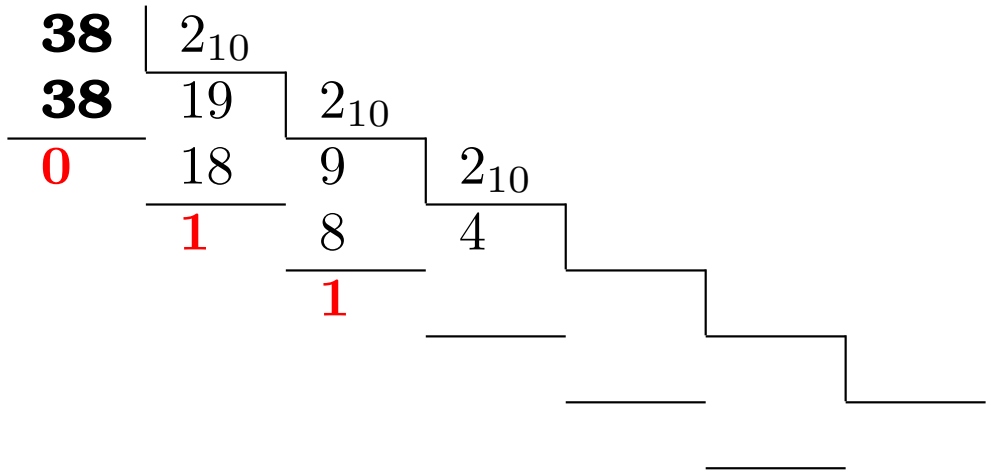

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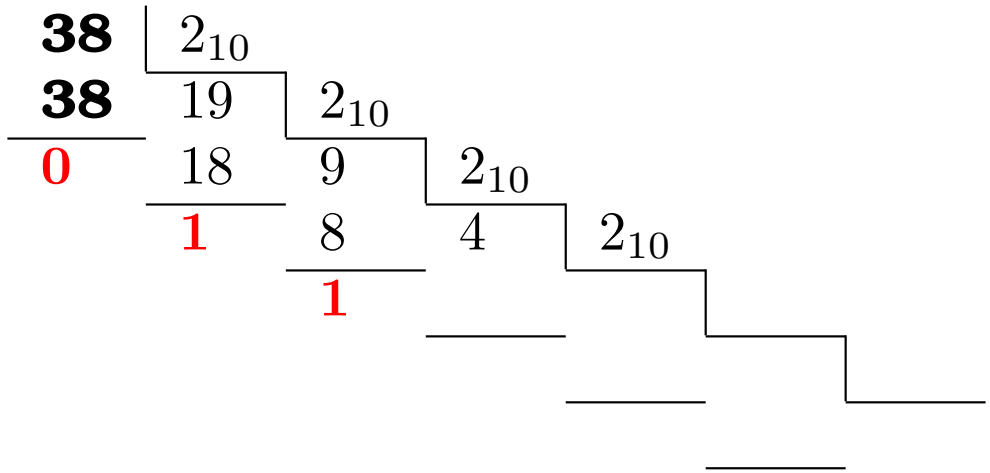

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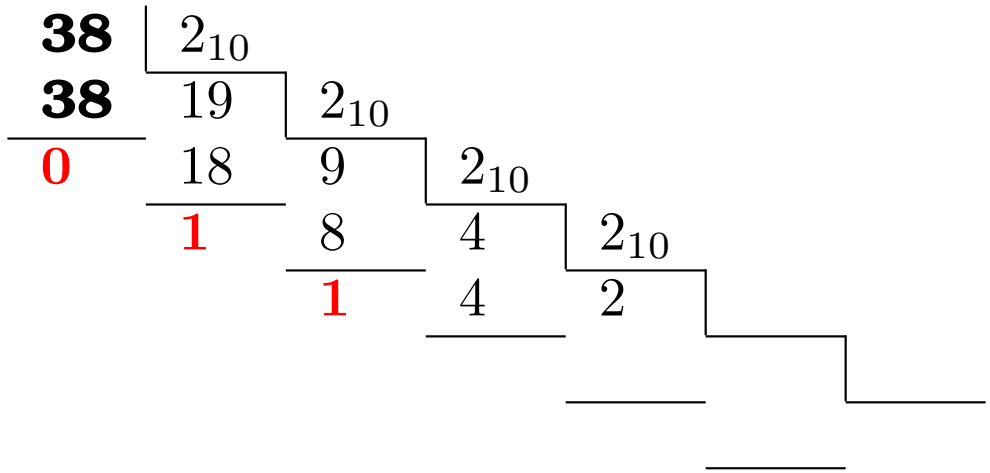

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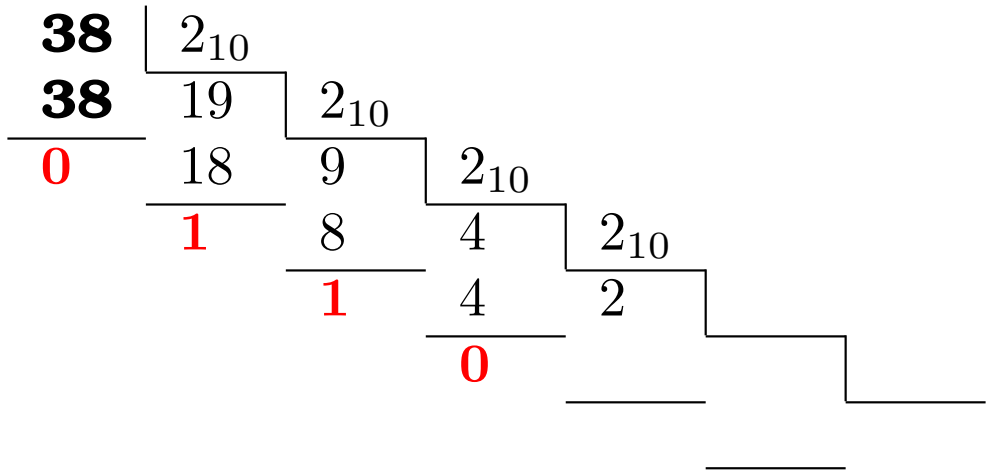

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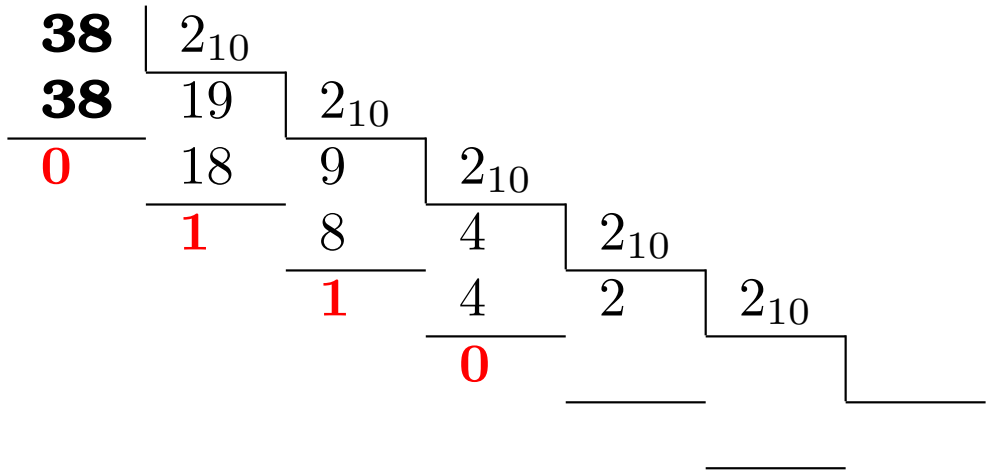

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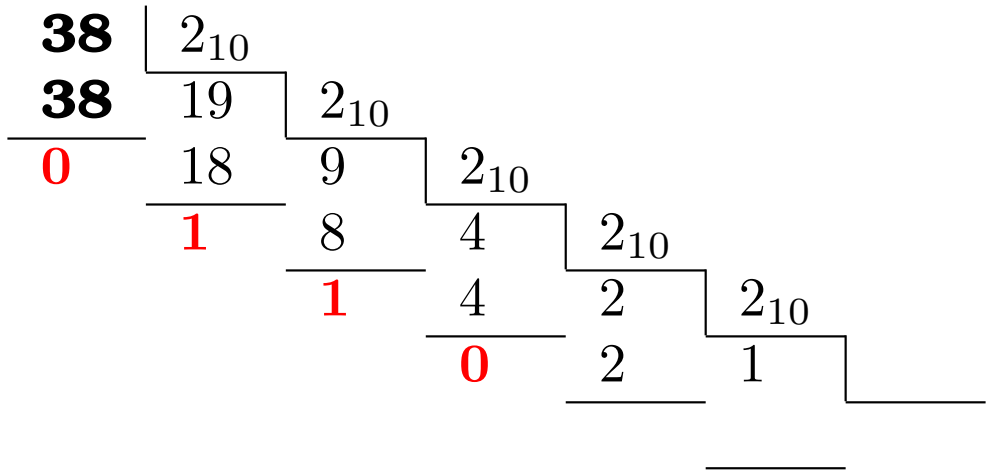

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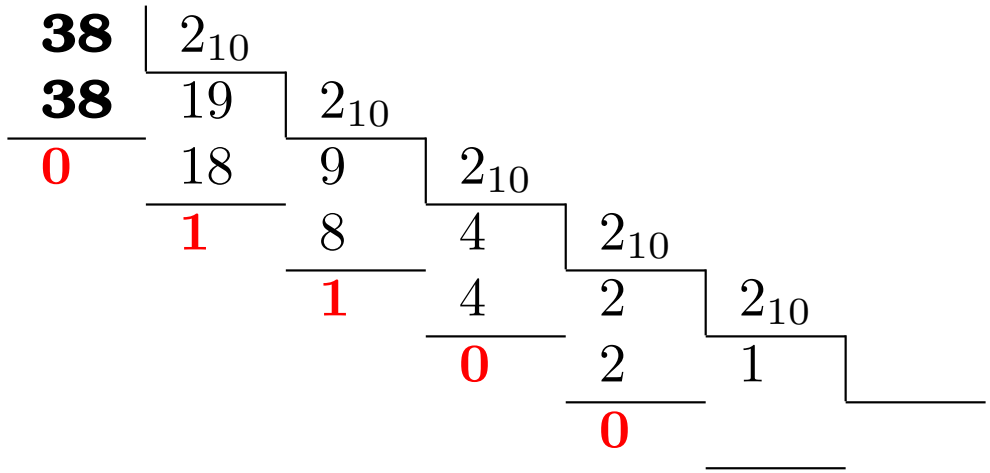

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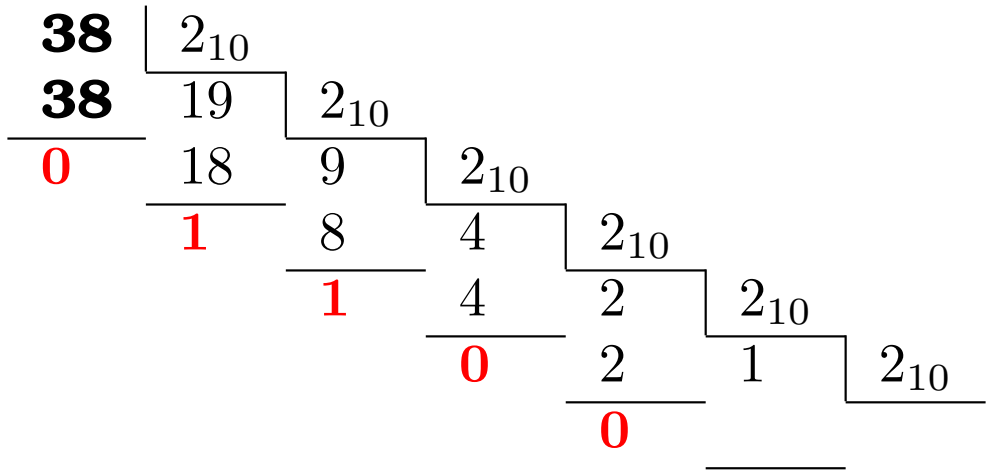

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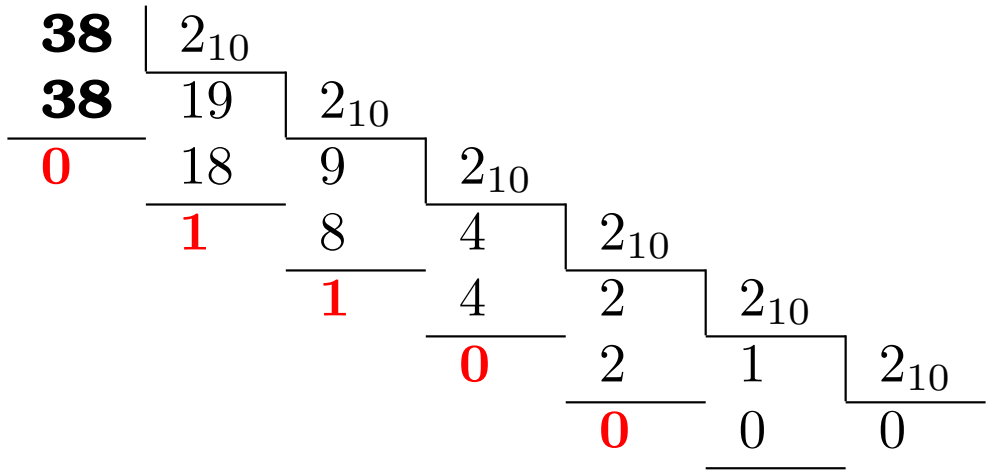

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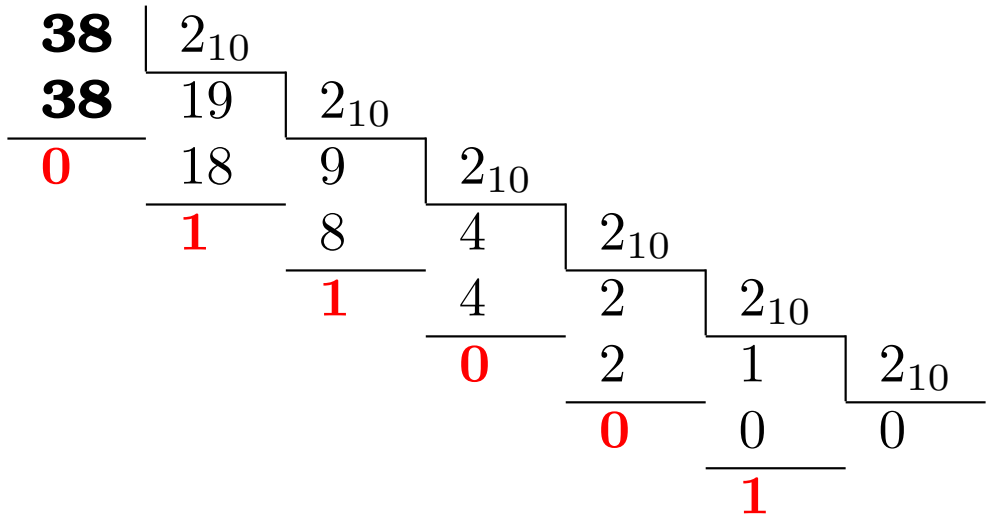

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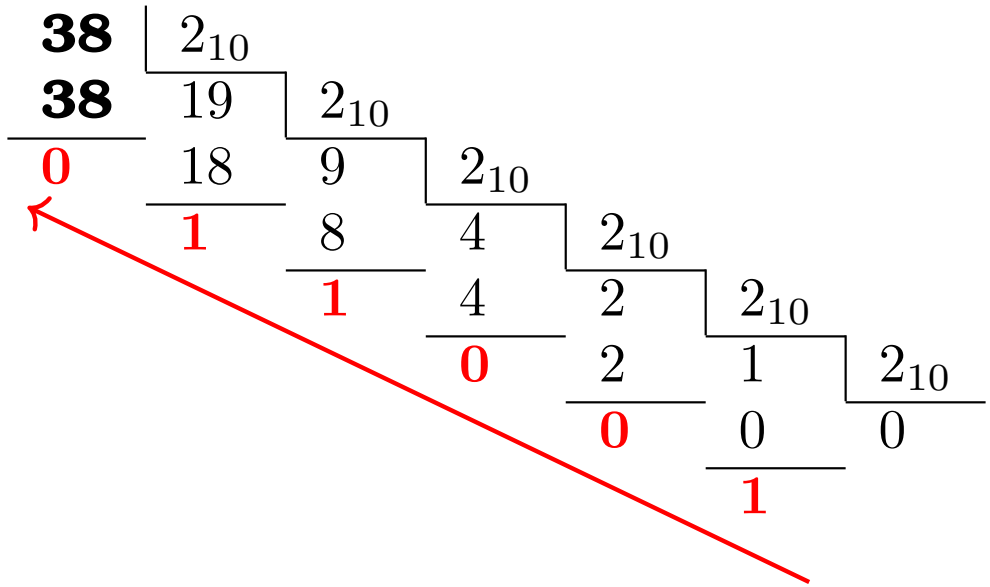

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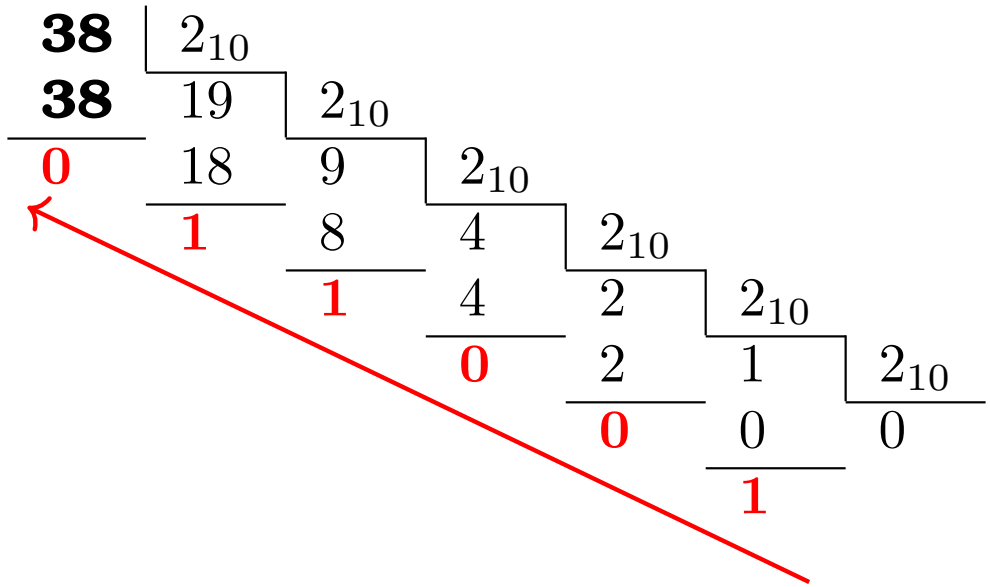

$$100110_2 = 2^5 + 2^2 + 2^1 = 32_{10} + 4_{10} + 2_{10} = \mathbf{38_{10}}$$

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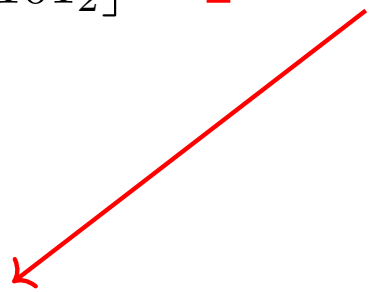

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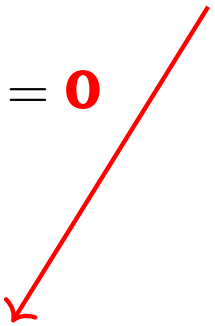

0.101_2

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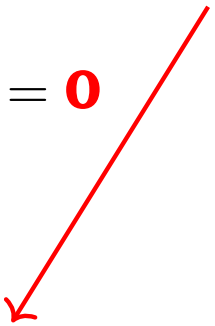

$$0.101_2 = 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} = 0.5_{10} + 0.125_{10} = \mathbf{0.625_{10}}$$

Converting to/from base 8 and base 16

Dividing by 8_{10} gives the **3** last binary digits as a reminder:

$$1101\,1011_2/8_{10} \equiv 1101\,1011_2/1000_2 = 1101\,1.\mathbf{011}_2$$

Dividing by 16_{10} gives the **4** last binary digits as a reminder:

$$1101\,1011_2/16_{10} \equiv 1101\,1011_2/10000_2 = 1101\,.\mathbf{1011}_2$$

Therefore:

To convert to octal (hexadecimal), group your binary digits by 3 (4) and convert each group to a corresponding digit:

Octal digits			Hex digits			
000	0		0000	0	1000	8
001	1		0001	1	1001	9
010	2		0010	2	1010	A
011	3		0011	3	1011	B
100	4		0100	4	1100	C
101	5		0101	5	1101	D
110	6		0110	6	1110	E
111	7		0111	7	1111	F

$\longleftarrow$

Example (octal):

$01011011_2 = \mathbf{001\ 011\ 011_2} = \mathbf{133_8}$

$\longrightarrow$

Example (hexadecimal):

$01011011_2 = \mathbf{0101\ 1011_2} = \mathbf{5B_{16}}$

XXI century: converting with computer

Manual:

```
man perlfaq4
```

Binary $\rightarrow$ decimal:

```
perl -le 'print 0b_0110_1011'
107
```

Hex $\rightarrow$ binary:

```
perl -e 'printf "%016b\n", 0x23FF'
0010001111111111
```

Octal $\rightarrow$ hex:

```
perl -e 'printf "%02X\n", 0133'
5B
```

Caution: this works as expected only for small numbers ($|n| < 2^{31}$)!

XXI century: converting fractions

Convert binary fraction $0.0110\,1011_2 = 0110\,1011_2/1\,0000\,0000_2$ to decimal:

```
perl -le 'print 0b_0110_1011 / (2**8)'
0.41796875
```

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```

Check it:

```
perl -e '$x=0.41796875; print "0."; while($x) {$x*=2; print int($x); $x-=int($x)}; print "\n"'
0.01101011
```

Caution: this method (and the check) works ***only*** if the fraction is **exactly** representable in our computer!

Convert 0.1_{10}

$$0.1_{10} = 1/10_{10}$$

File: `"examples/code/one-tenth.c"`

```c
#include <stdio.h>
#include <stdlib.h>

int main( int argc, char *argv[] )
{
    int n = 1;  // numerator
    int d = 10; // denominator
    int b = 2;  // the base of the target number system
    int s = 0;  // current step
    int maxstep = 20;

    /* Get the max. digit count: */
    printf( "%d.", n/d );
    while( n != 0 && s < maxstep ) {
        n *= b;
        printf( "%d", n/d ); //  |_ b*(n/d) _|
        n %= d;              //  b*(n/d) - |_ b*(n/d) _|
        s++;
        if( s % 4 == 0 ) {
            printf( " " );   // print space after each 4 digits
        }
    }
    printf( "\n" );

    return 0;
}
```

Convert 0.1_{10}

$$0.1_{10} = 1/10_{10}$$

```
sh$ examples/code/one-tenth
0.0001 1001 1001 1001 1001
```

$$0.1_{10} = 0.0001\,\overline{1001}_2$$

Infinite periodic (repeating) binary fraction!

If the fully reduced denominator of a fraction has a prime factor that is ***not*** a factor of a number base, then expansion is non-terminating in that base.

E.g. $1/10_{10}$:

$$10_{10} = 2 \times 5; \quad 5 \text{ is not a factor of } 2$$
$$\Rightarrow \quad 1/10_{10} = 0.0001\,\overline{1001}_2 \text{ is non-terminating}$$

Binary arithmetic

/.../ Būtinas minimumas yra sudėtis, nes atimtis tai yra sudėtis su papildomu kodu, daugyba – daug sudėčių cikle, o dalyba – daug atimčių.

The single required operation is addition, since subtraction is addition in complementary code, multiplication is multiple additions in a loop and division is multiple subtractions.

Antanas Mitašiūnas (2016)
"Kompiuterių architektūra", p. 9

Addition:

$$A + B$$

		B	
	+	0	1
A	0	0	1
	1	1	10

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Addition:

$A + B$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$A \times B$

A \ B	0	1
0	0	0
1	0	1

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{r} 11101 \\ + \quad 11001 \\ \hline \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & {\scriptstyle 1} \\ + & 11101 \\ & 11001 \\ \hline & 0 \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & 1 \\ & 11101 \\ + & 11001 \\ \hline & 10 \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & 1 \\ & 11101 \\ + & 11001 \\ \hline & 110 \end{array}$$

Binary arithmetic

Addition:

$A + B$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$A \times B$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & {\scriptstyle 1\quad 1} \\ & 11101 \\ + & 11001 \\ \hline & 0110 \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & {\scriptstyle 1\,1} \quad {\scriptstyle 1} \\ & 1\,1\,1\,0\,1 \\ + & 1\,1\,0\,0\,1 \\ \hline & 1\,0\,1\,1\,0 \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ +	B: 0	B: 1
0	0	1
1	1	10

Multiplication:

$$A \times B$$

A \ ×	B: 0	B: 1
0	0	0
1	0	1

$$\begin{array}{r} {\scriptstyle 11\quad 1} \\ 11101 \\ + \quad 11001 \\ \hline 110110 \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

$$\begin{array}{rr} & {\scriptstyle 1\,1} \quad {\scriptstyle 1} \\ & 1\,1\,1\,0\,1 \\ + & 1\,1\,0\,0\,1 \\ \hline & 1\,1\,0\,1\,1\,0 \end{array}$$

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & 1\,1\,0\,1 \\ \times & 1\,0\,1 \\ \hline & \\ & \\ & \\ \hline & \\ & \\ \hline \end{array}$$

Binary arithmetic

Addition:

$A + B$

A \ B	0	1
0	0	1
1	1	10

$$\begin{array}{r} {\scriptstyle 1\,1\quad\ 1} \\ 11101 \\ +\ 11001 \\ \hline 110110 \end{array}$$

Multiplication:

$A \times B$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{r} 1101 \\ \times\quad 101 \\ \hline 1101 \\ \\ \hline \end{array}$$

Binary arithmetic

Addition:

$$A + B$$

A \ B	0	1
0	0	1
1	1	10

$$\begin{array}{rr} & {\scriptstyle 1\,1} \quad {\scriptstyle 1} \\ & 1\,1\,1\,0\,1 \\ + & 1\,1\,0\,0\,1 \\ \hline & 1\,1\,0\,1\,1\,0 \end{array}$$

Multiplication:

$$A \times B$$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{rr} & 1101 \\ \times & 101 \\ \hline & \\ & 1101 \\ & 0000 \\ & \\ \hline & \end{array}$$

Binary arithmetic

Addition:

$A + B$

A \ B	0	1
0	0	1
1	1	10

$$\begin{array}{r} {\scriptstyle 1\,1\quad\;\,1} \\ 11101 \\ +\;\;11001 \\ \hline 110110 \end{array}$$

Multiplication:

$A \times B$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{r} 1101 \\ \times\quad\;\;101 \\ \hline 1101 \\ 0000 \\ 1101 \\ \hline \end{array}$$

Binary arithmetic

Addition:

$A + B$

A \ B	0	1
0	0	1
1	1	10

$$\begin{array}{r} {\scriptstyle 1\,1\quad\ 1} \\ 11101 \\ +\ 11001 \\ \hline 110110 \end{array}$$

Multiplication:

$A \times B$

A \ B	0	1
0	0	0
1	0	1

$$\begin{array}{r} 1101 \\ \times\quad 101 \\ \hline 111 \\ 1101 \\ 0000 \\ 1101 \\ \hline 1000001 \end{array}$$

Bitwise logic operations

We will abbreviate the term *binary digit* to *bit*.

Useful operations can be performed on the single bits of a number independently.

Mask bits:

$$\begin{array}{rr} & 1101\,1010\,0111 \\ \text{AND} & 0001\,1111\,1000 \\ \hline & 0001\,1010\,0000 \end{array}$$

Bitwise logic operations

We will abbreviate the term *binary digit* to *bit*.

Useful operations can be performed on the single bits of a number independently.

Mask bits:

```
      1101 1010 0111
AND   0001 1111 1000
-----------------------
      0001 1010 0000
```

Set bits:

```
      1101 1010 0111
OR    0001 1111 1000
-----------------------
      1101 1111 1111
```

Bitwise logic operations

We will abbreviate the term *binary digit* to *bit*.

Useful operations can be performed on the single bits of a number independently.

Mask bits:

$$\begin{array}{rr} & 1101\,1010\,0111 \\ \text{AND} & 0001\,1111\,1000 \\ \hline & 0001\,1010\,0000 \end{array}$$

Set bits:

$$\begin{array}{rr} & 1101\,1010\,0111 \\ \text{OR} & 0001\,1111\,1000 \\ \hline & 1101\,1111\,1111 \end{array}$$

Questions:

$$2\text{B OR NOT } 2\text{B} = ??$$

Bitwise logic operations

We will abbreviate the term *binary digit* to *bit*.

Useful operations can be performed on the single bits of a number independently.

Mask bits:

$$\begin{array}{rr} & 1101\,1010\,0111 \\ \text{AND} & 0001\,1111\,1000 \\ \hline & 0001\,1010\,0000 \end{array}$$

Set bits:

$$\begin{array}{rr} & 1101\,1010\,0111 \\ \text{OR} & 0001\,1111\,1000 \\ \hline & 1101\,1111\,1111 \end{array}$$

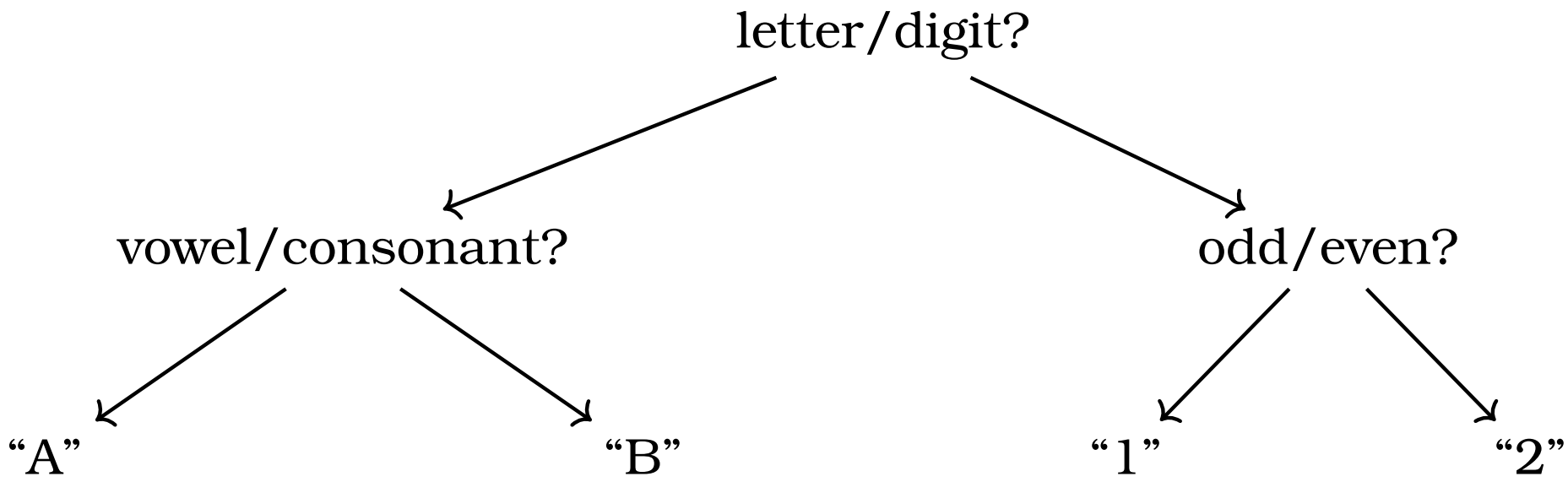
Questions:

$$2\text{B OR NOT } 2\text{B} = ??$$

What is the bit mask to check whether the number is odd?

Intuitive understanding of information content

Find out which of four symbols was transmitted by asking yes/no questions: “A”, “B”, “1”, “2”.



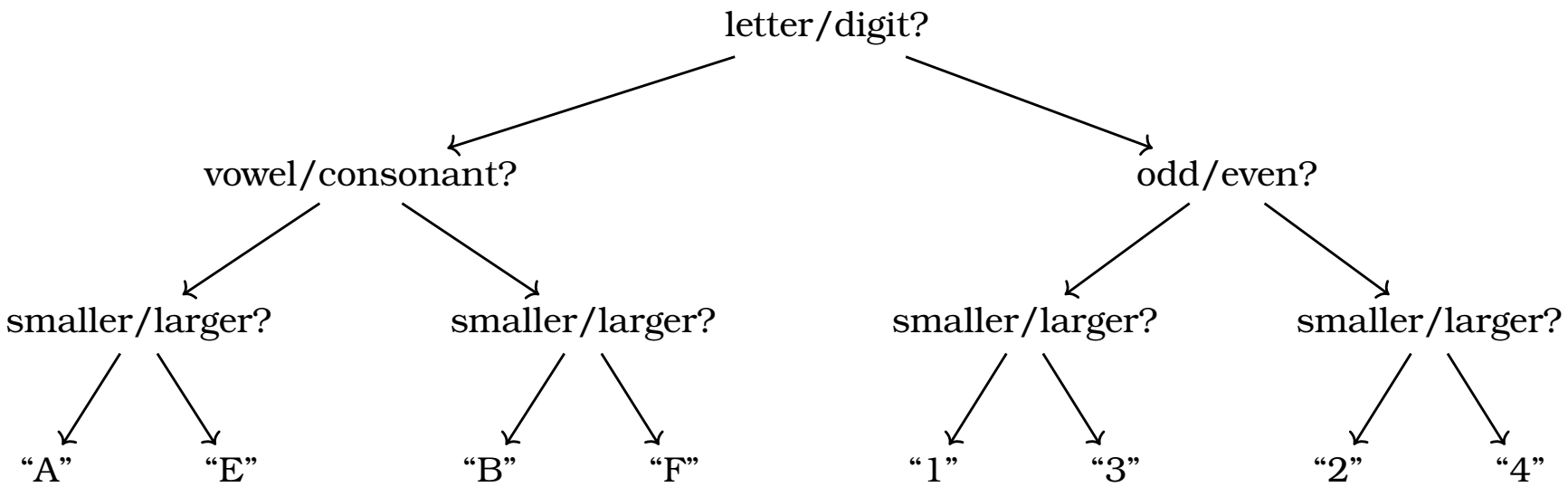
$$N_{\text{symbols}} = N_{\text{leaves}} = 2^{n_{\text{questions}}} = 2^{n_{\text{layers}}}$$

$$n_{\text{questions}} = n_{\text{layers}} = \log_2 N_{\text{symbols}} = \log_2 \frac{1}{p_{\text{symbol}}} = -\log_2 p_{\text{symbol}}$$

$$n_{\text{q}} = \log_2 N_{\text{s}} = \log_2 4 = -\log_2 \tfrac{1}{4} = \log_2 2^2 = 2$$

Intuitive understanding of information (2)

One out of 8 symbols: “A”, “B”, “E”, “F”, “1”, “2”, “3”, “4”.



$$N_{\text{symbols}} = N_{\text{leaves}} = 2^{n_{\text{questions}}} = 2^{n_{\text{layers}}}$$

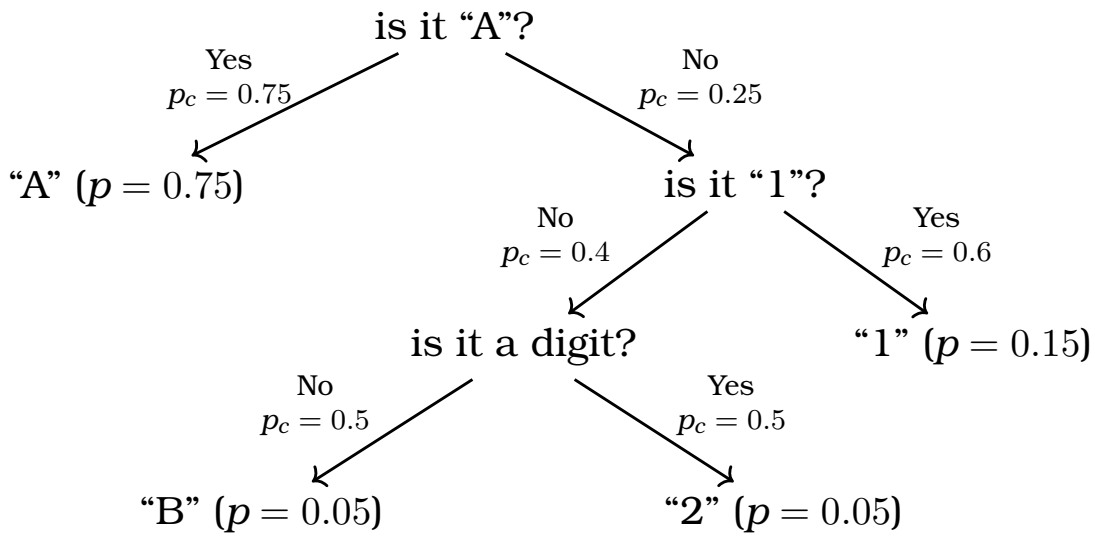
$$n_{\text{questions}} = n_{\text{layers}} = \log_2 N_{\text{symbols}} = \log_2 \frac{1}{p_{\text{symbol}}} = -\log_2 p_{\text{symbol}}$$

$$n_{\text{q}} = \log_2 N_{\text{s}} = \log_2 8 = -\log_2 \tfrac{1}{8} = \log_2 2^3 = 3$$

Info content depends in probability

One out of 4 symbols: “A”, “B”, “1”, “2”

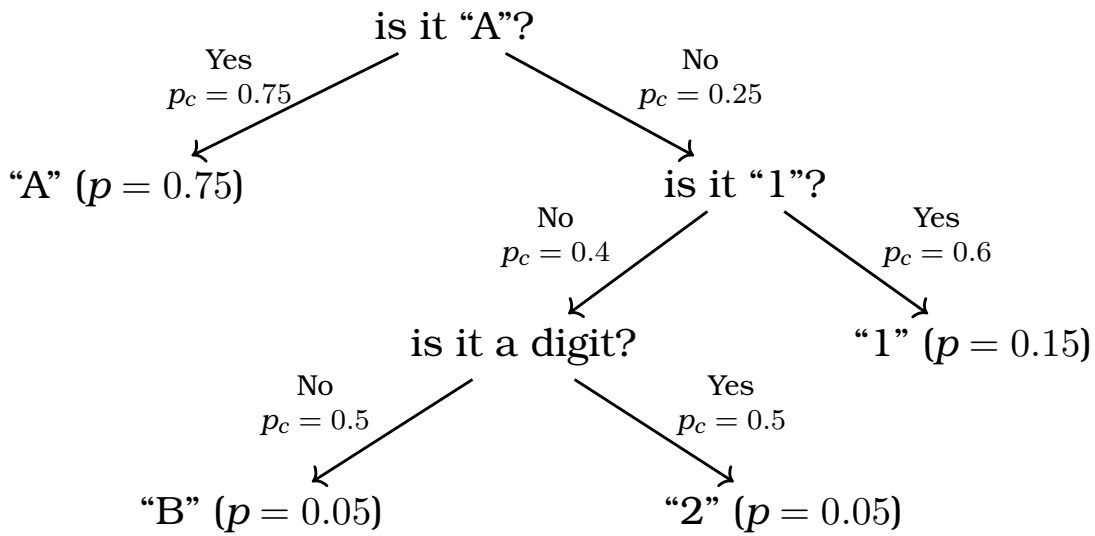
“AAA1AAABA1A2AAA1AAAA”
(20 symbols)



Info content depends in probability

One out of 4 symbols: “A”, “B”, “1”, “2”

“AAA1AAABA1A2AAA1AAAA”
(20 symbols)

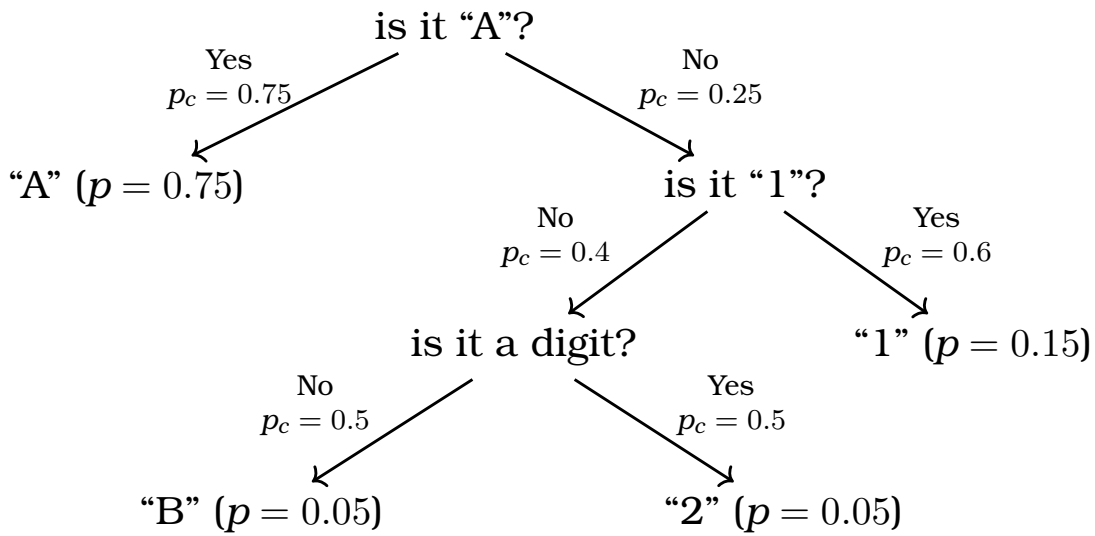


For equal probabilities: $n_{\mathrm{q}} = \log_2 N_{\mathrm{s}} = \log_2 \frac{1}{p_{\mathrm{s}}} = -\log_2 p_{\mathrm{s}} = \log_2 4 = \mathbf{2}$

Info content depends in probability

One out of 4 symbols: “A”, “B”, “1”, “2”

“AAA1AAABA1A2AAA1AAAA”
(20 symbols)

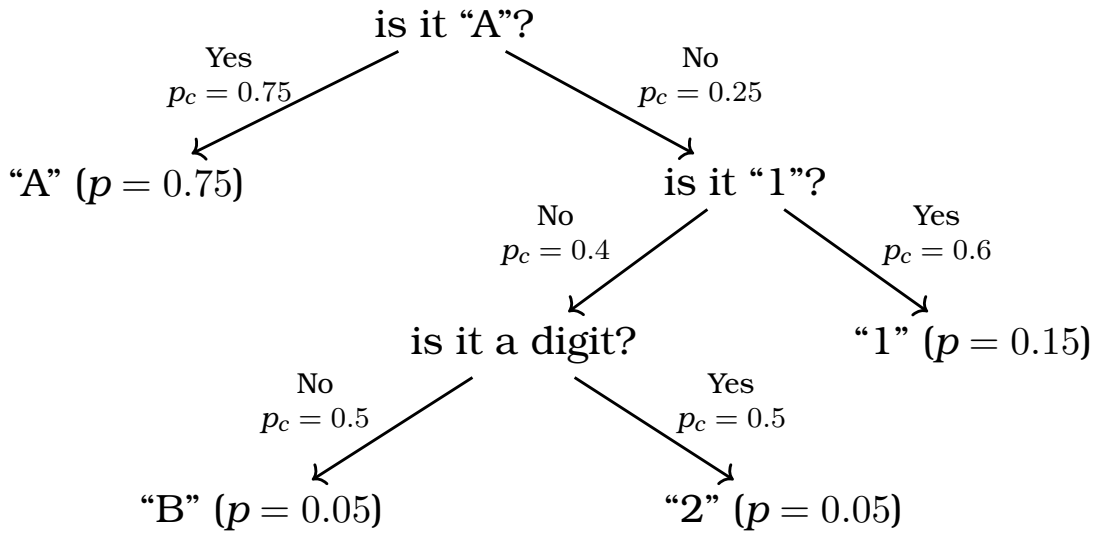


For equal probabilities: $n_\mathrm{q} = \log_2 N_\mathrm{s} = \log_2 \frac{1}{p_\mathrm{s}} = -\log_2 p_\mathrm{s} = \log_2 4 = \mathbf{2}$
For the specified question sequence (Huffman encoding):
$n_\mathrm{q} = 0.75 + 2 \times 0.15 + 2 \times 3 \times 0.05 = \mathbf{1.35}$

Info content depends in probability

One out of 4 symbols: “A”, “B”, “1”, “2”

“AAA1AAABA1A2AAA1AAAA”
(20 symbols)



For equal probabilities: $n_\mathrm{q} = \log_2 N_\mathrm{s} = \log_2 \frac{1}{p_\mathrm{s}} = -\log_2 p_\mathrm{s} = \log_2 4 = \mathbf{2}$
For the specified question sequence (Huffman encoding):
$n_\mathrm{q} = 0.75 + 2 \times 0.15 + 2 \times 3 \times 0.05 = \mathbf{1.35}$
Theoretical limit: $n_\mathrm{q} = -\sum_{i=1}^{4} p_i \log_2 p_i = \mathbf{1.15}$

Bits and bytes. Information content

Assume we have a source that can send messages from the set $S = \{M_1, \ldots, M_n\},\ n \in \mathbb{N}$.
Then:

$$S = \{M_1, \ldots, M_n\}$$

$$p_i = Pr(M_i), \quad i = 1 \ldots n$$

$$I_{M_i} = -\log p_i$$

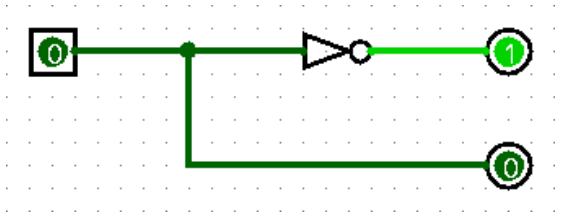
$$H = -\sum_{i=1}^{n} p_i \log p_i$$

Here:

$Pr(M_i)$	–	probability of message M_i;
I_{M_i}	–	information content of message M_i;
H	–	entropy of the source, or average information content per message

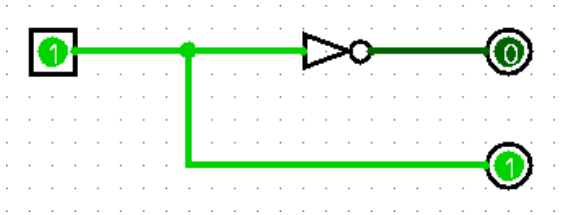
Two level signal $\neq$ 1 bit

Two-wire single bit representation:



Two level signal $\neq$ 1 bit

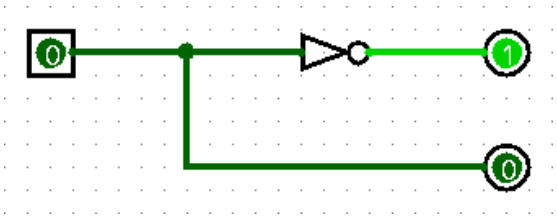
Two-wire single bit representation:



Two level signal $\neq$ 1 bit

$$
\begin{aligned}
Pr(\text{"0"}) &= Pr(\text{"1"}) = 0.5 \\
H &= -2 \cdot 0.5 \cdot \log_2 0.5 \\
&= -2 \cdot (-0.5) \\
&= \textbf{1 bit/message}
\end{aligned}
$$

$$
\begin{aligned}
Pr(\text{"01"}) &= Pr(\text{"10"}) = 0.5 \\
Pr(\text{"00"}) &= Pr(\text{"11"}) = 0 \\
H &= - Pr(\text{"01"}) \cdot \log_2 Pr(\text{"01"}) \\
&\quad - Pr(\text{"10"}) \cdot \log_2 Pr(\text{"10"}) \\
&\quad - \lim_{Pr(\text{"00"}) \to 0} Pr(\text{"00"}) \cdot \log_2 Pr(\text{"00"}) \\
&\quad - \lim_{Pr(\text{"11"}) \to 0} Pr(\text{"11"}) \cdot \log_2 Pr(\text{"11"}) \\
&= - 2 \cdot 0.5 \cdot (-1) \\
&= \textbf{1 bit/message}
\end{aligned}
$$



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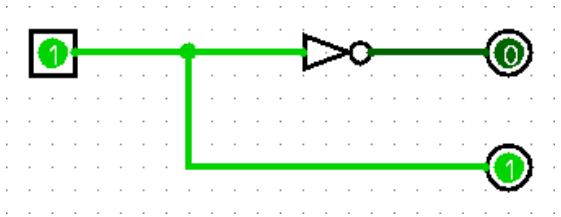
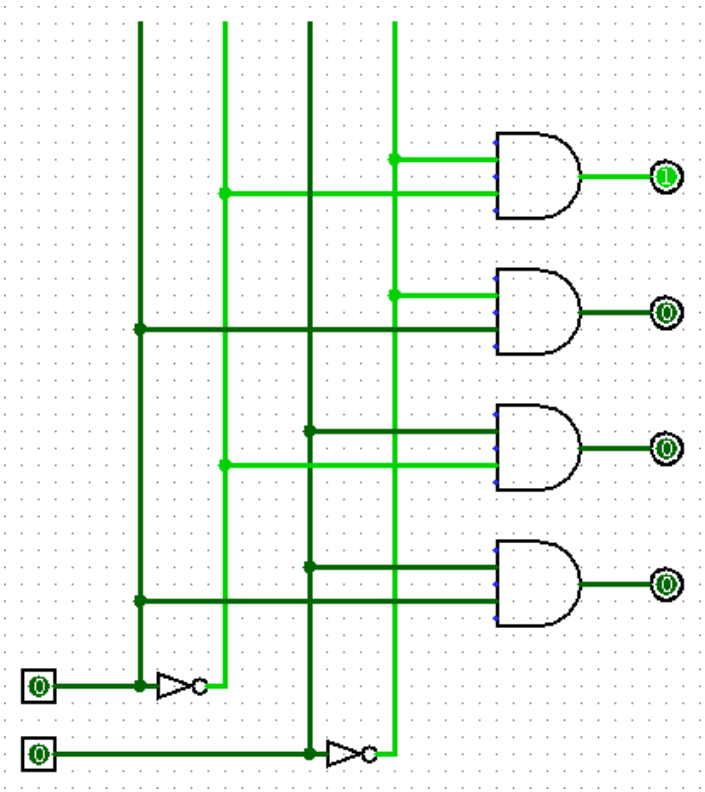
&= -2 \cdot (-0.5) \\
&= \textbf{1 bit/message}
\end{aligned}
$$

$$
\begin{aligned}
Pr(\text{"01"}) &= Pr(\text{"10"}) = 0.5 \\
Pr(\text{"00"}) &= Pr(\text{"11"}) = 0 \\
H &= - Pr(\text{"01"}) \cdot \log_2 Pr(\text{"01"}) \\
&\quad - Pr(\text{"10"}) \cdot \log_2 Pr(\text{"10"}) \\
&\quad - \lim_{Pr(\text{"00"}) \to 0} Pr(\text{"00"}) \cdot \log_2 Pr(\text{"00"}) \\
&\quad - \lim_{Pr(\text{"11"}) \to 0} Pr(\text{"11"}) \cdot \log_2 Pr(\text{"11"}) \\
&= - 2 \cdot 0.5 \cdot (-1) \\
&= \textbf{1 bit/message}
\end{aligned}
$$



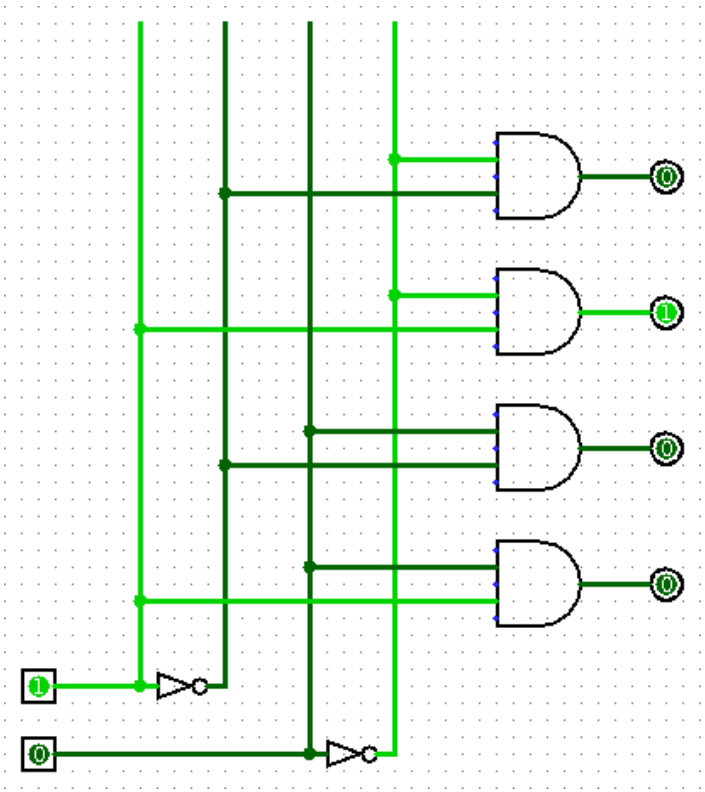
Two level signal $\neq$ 1 bit

Decoder 1-of-4



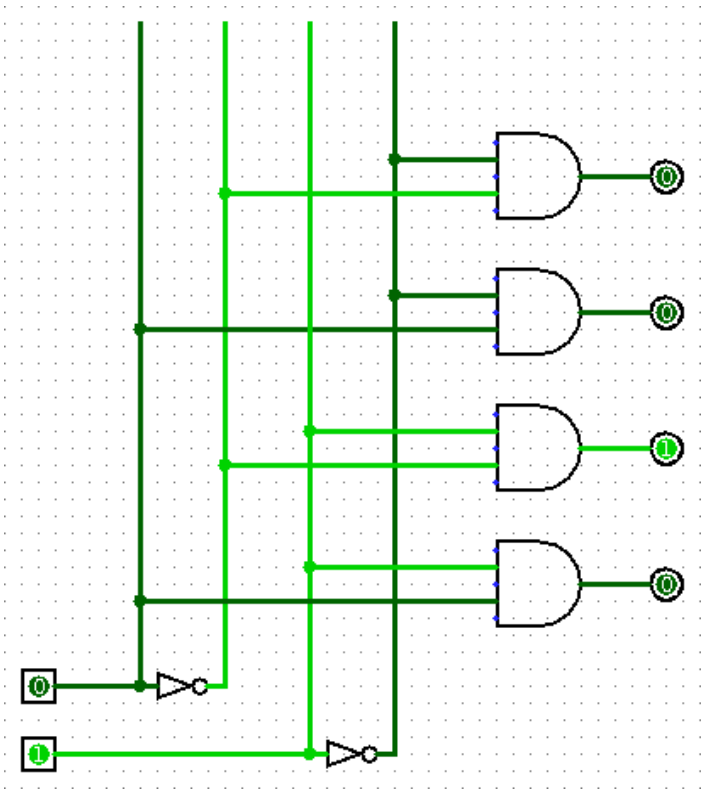
Two level signal $\neq$ 1 bit

Decoder 1-of-4



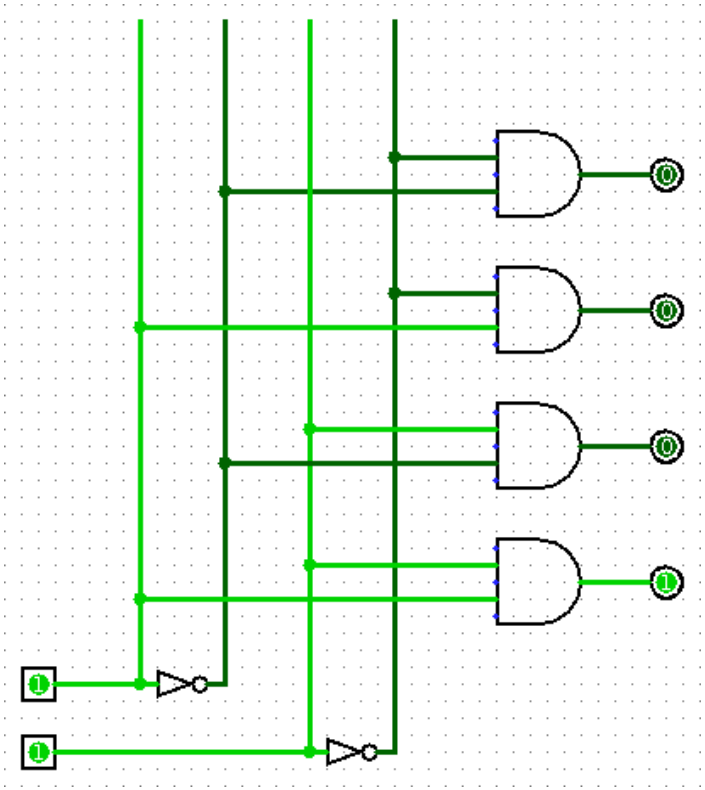
Two level signal $\neq$ 1 bit

Decoder 1-of-4



Two level signal $\neq$ 1 bit

Decoder 1-of-4



Half-adder

Addition:

$A + B$

A \ +	B: 0	B: 1
0	0	1
1	1	10

Sum:

A	**B**	**S**
0	0	0
0	1	1
1	0	1
1	1	0

Carry:

A	**B**	**C**
0	0	0
0	1	0
1	0	0
1	1	1

Half-adder

Addition:

$A + B$

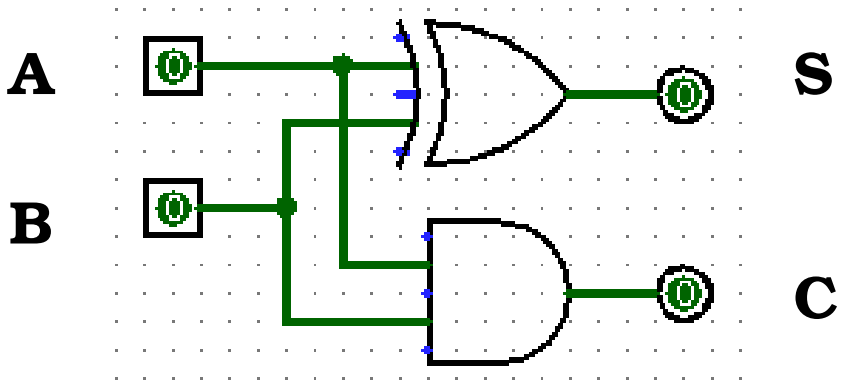
A \ +	B = 0	B = 1
0	0	1
1	1	10

Sum:

A	B	S
0	0	0
0	1	1
1	0	1
1	1	0

Carry:

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1



Half-adder

Addition:

$A + B$

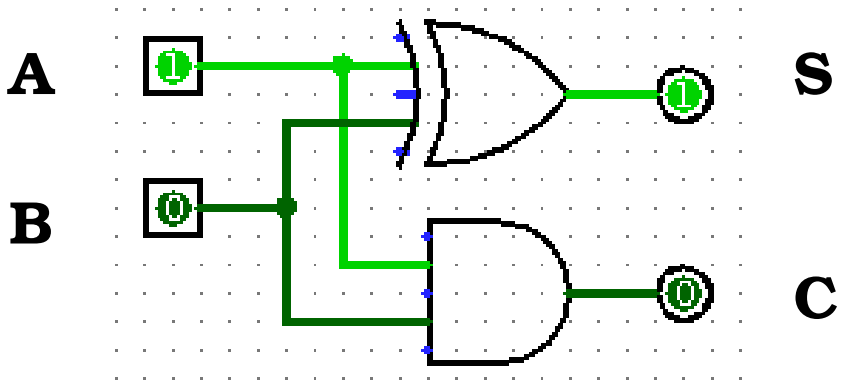
A \ B	0	1
0	0	1
1	1	10

Sum:

A	B	S
0	0	0
0	1	1
1	0	1
1	1	0

Carry:

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1



Half-adder

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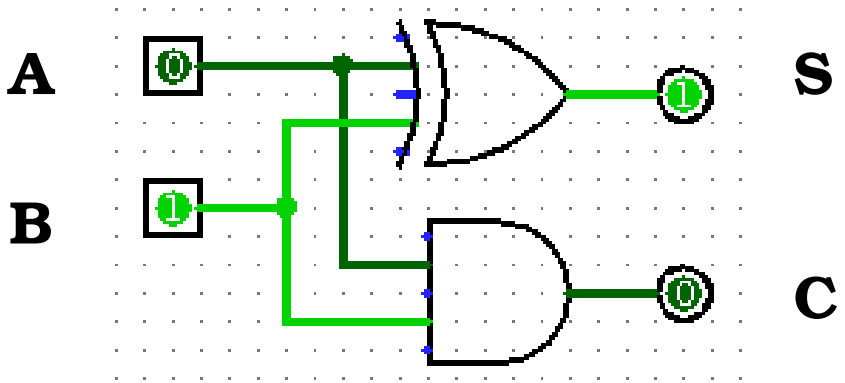
A \ B	0	1
0	0	1
1	1	10

Sum:

A	B	S
0	0	0
0	1	1
1	0	1
1	1	0

Carry:

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1



Half-adder

Addition:

$A + B$

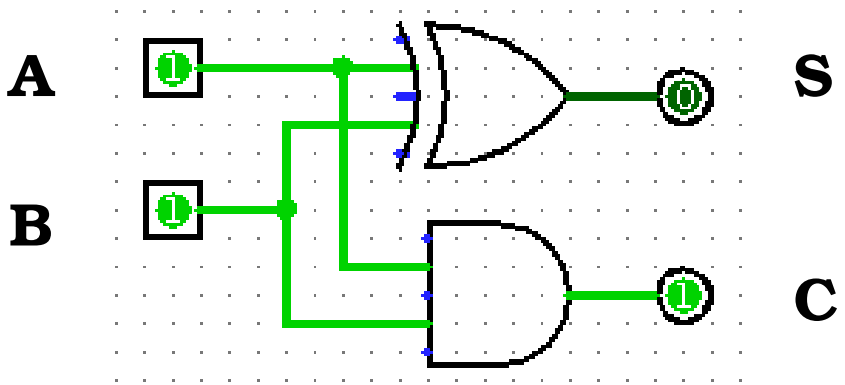
A \ +	B = 0	B = 1
0	0	1
1	1	10

Sum:

A	B	S
0	0	0
0	1	1
1	0	1
1	1	0

Carry:

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1



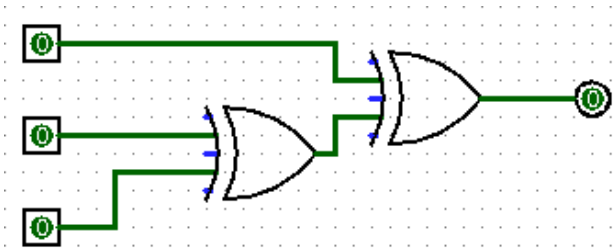
Full adder

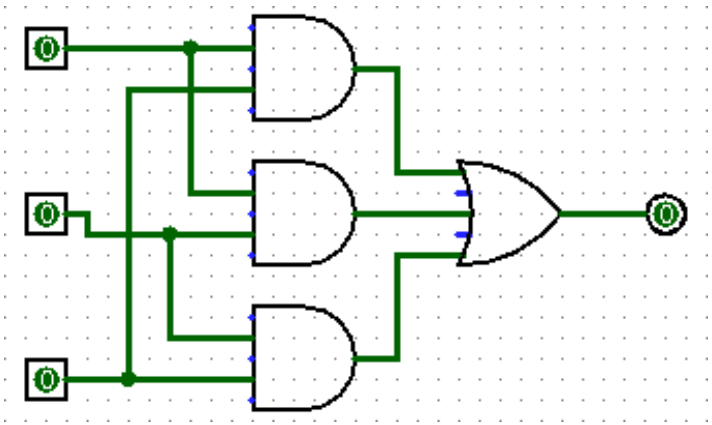
Sum:

C_{in}	A	B	S
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Carry:

C_{in}	A	B	C_{out}		
0	0	0	0		
0	0	1	0		
0	1	0	0		
0	1	1	1	$\overline{C_{in}}AB$	AB
1	0	0	0		
1	0	1	1	$C_{in}\overline{A}B$	$C_{in}B$
1	1	0	1	$C_{in}A\overline{B}$	$C_{in}A$
1	1	1	1	$C_{in}AB$	

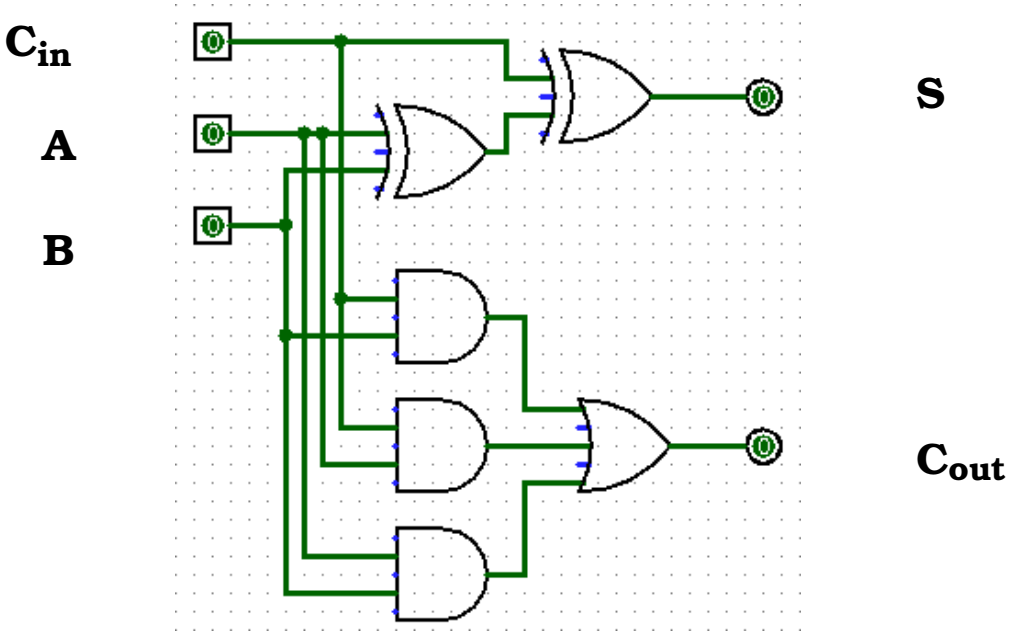




Full adder (naïve)

Sum and Carry:

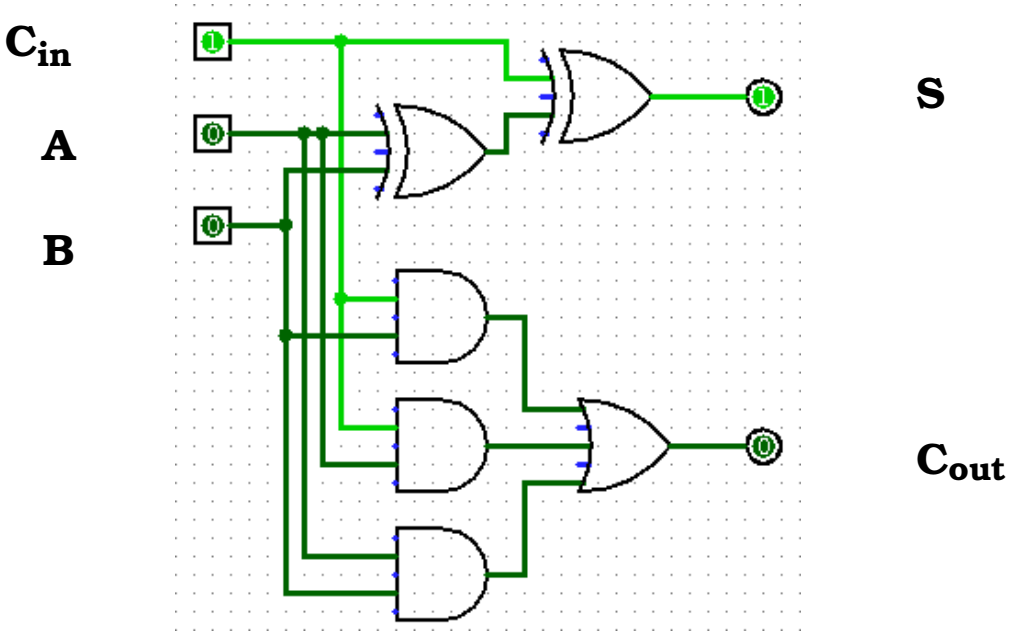
C_{in}	A	B	C_{out}	S
0	**0**	**0**	**0**	**0**
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



Full adder (naïve)

Sum and Carry:

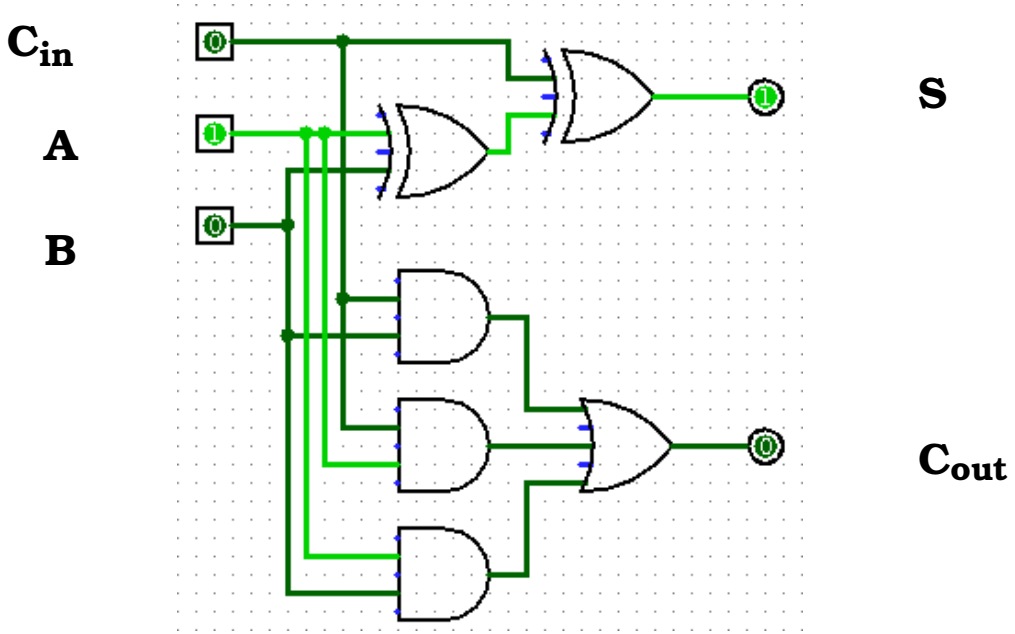
C_{in}	A	B	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	**0**	**0**	**0**	**1**
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



Full adder (naïve)

Sum and Carry:

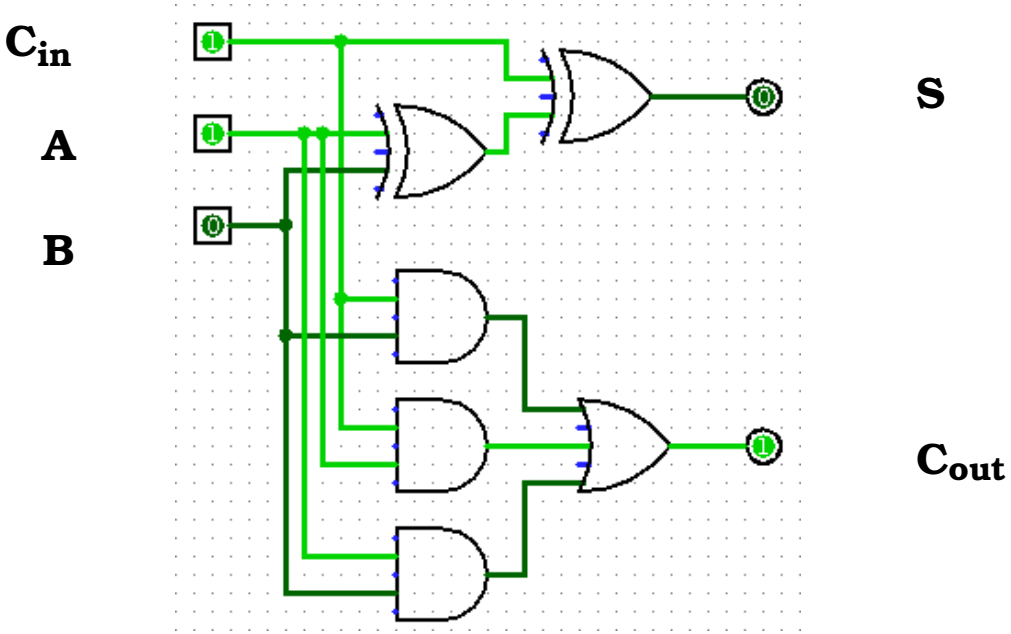
C_{in}	A	B	C_{out}	S
0	0	0	0	0
0	**0**	**1**	**0**	**1**
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



Full adder (naïve)

Sum and Carry:

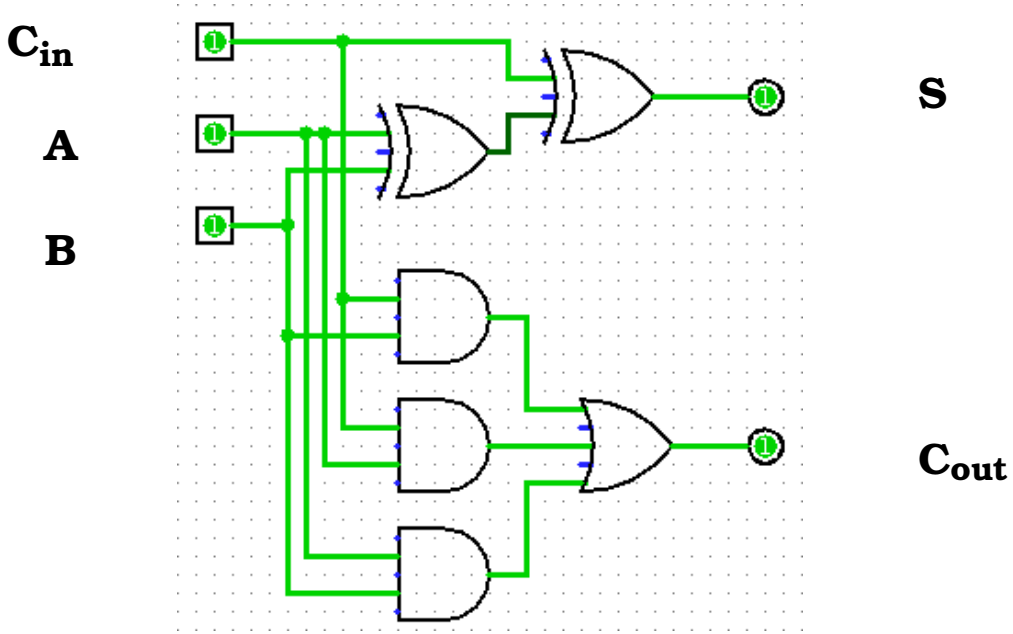
C_{in}	A	B	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	**0**	**1**	**1**	**0**
1	1	0	1	0
1	1	1	1	1



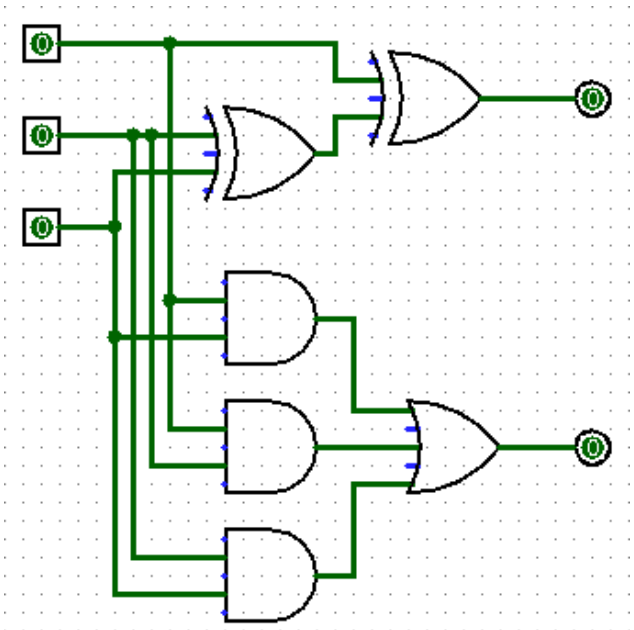
Full adder (naïve)

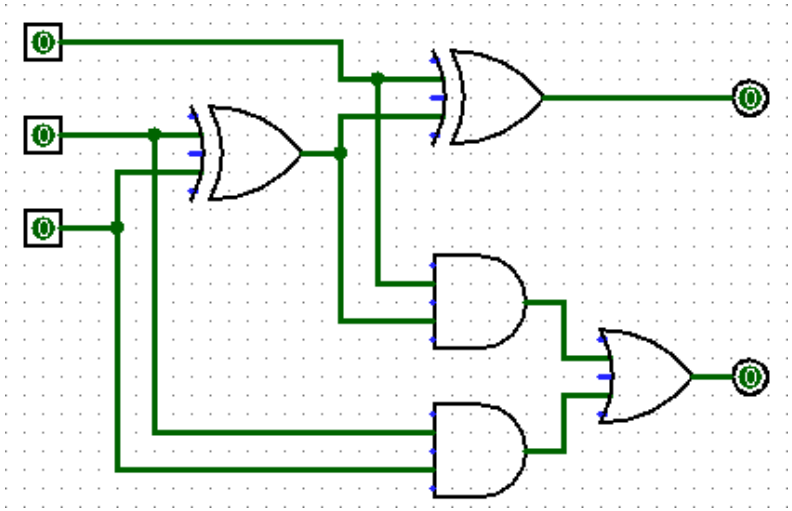
Sum and Carry:

C_{in}	A	B	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	**1**	**1**	**1**	**1**

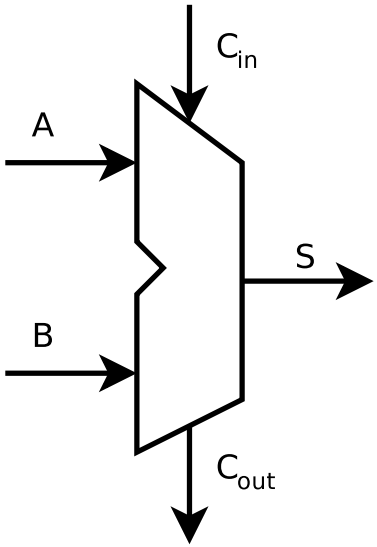


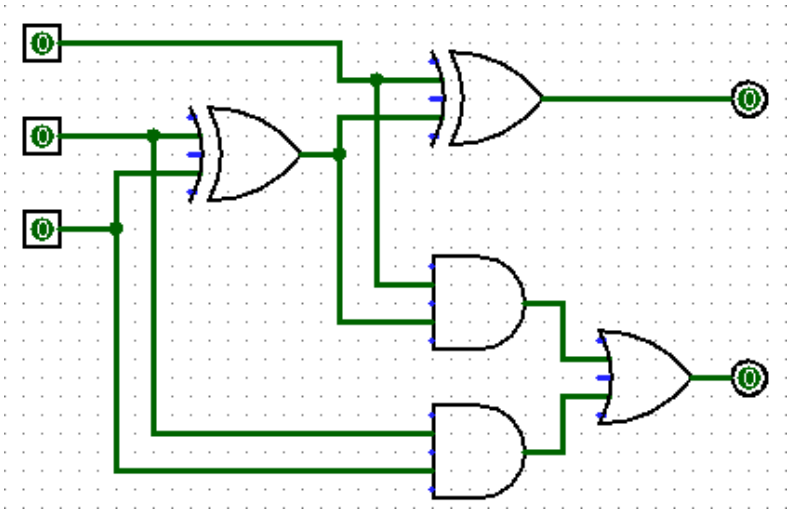
Full adder

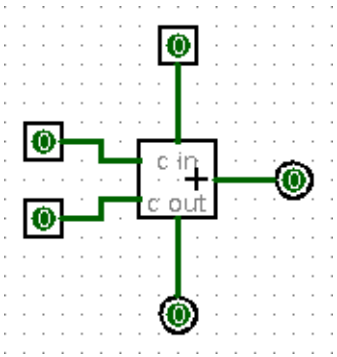




Full adder symbols

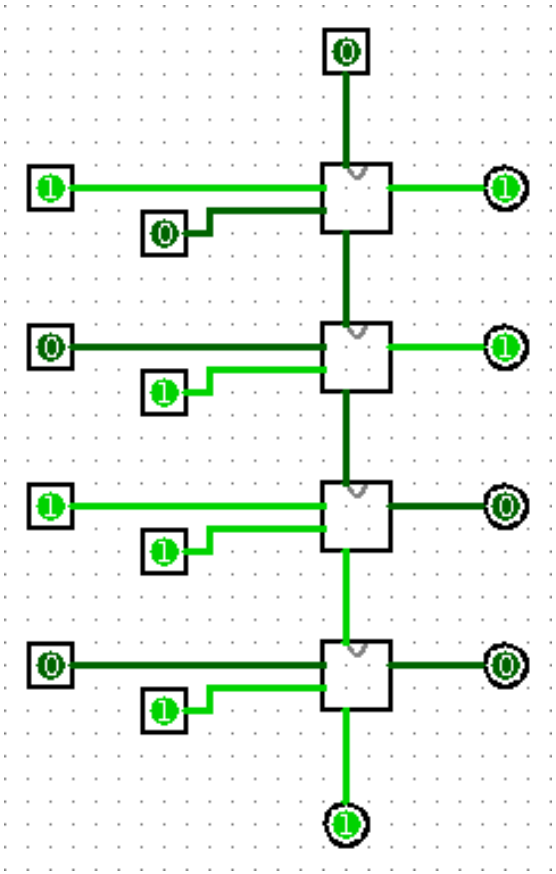


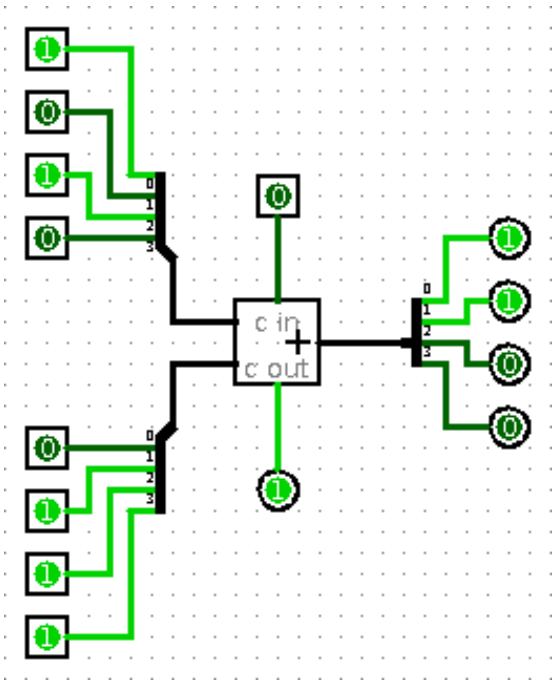




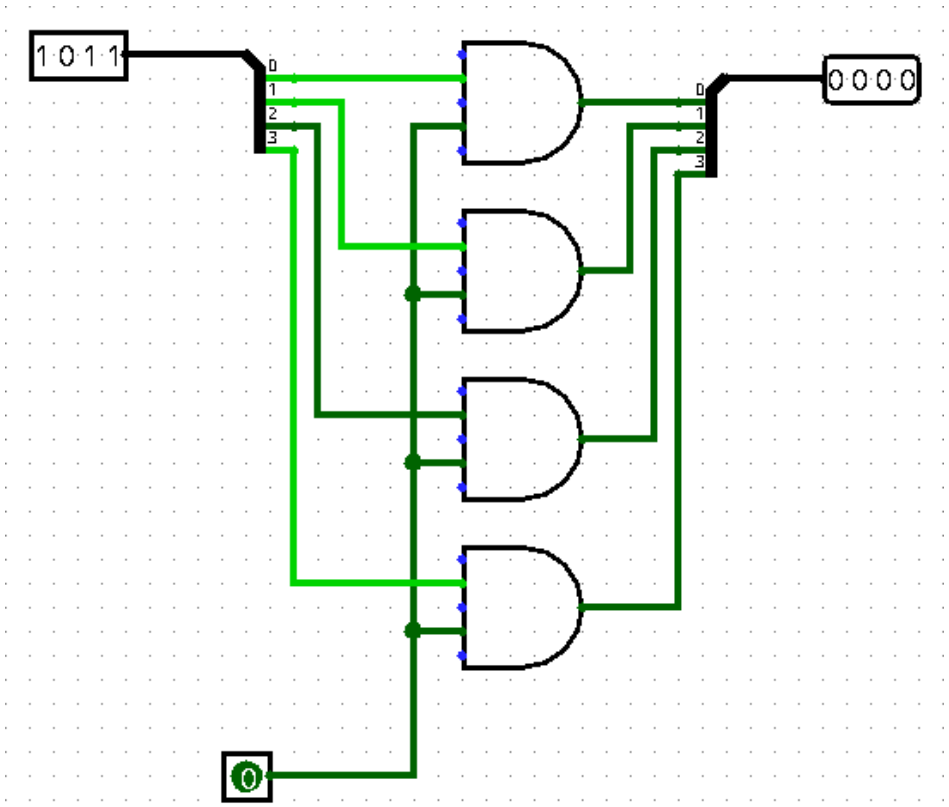
4-bit adder

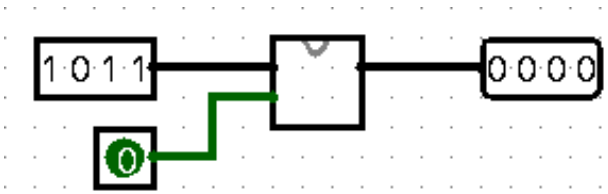
$$0101_2 + 1110_2 = 1\,0011_2 \qquad 5_{16} + \mathrm{E}_{16} = 13_{16} \qquad 5_{10} + 14_{10} = 19_{10}$$



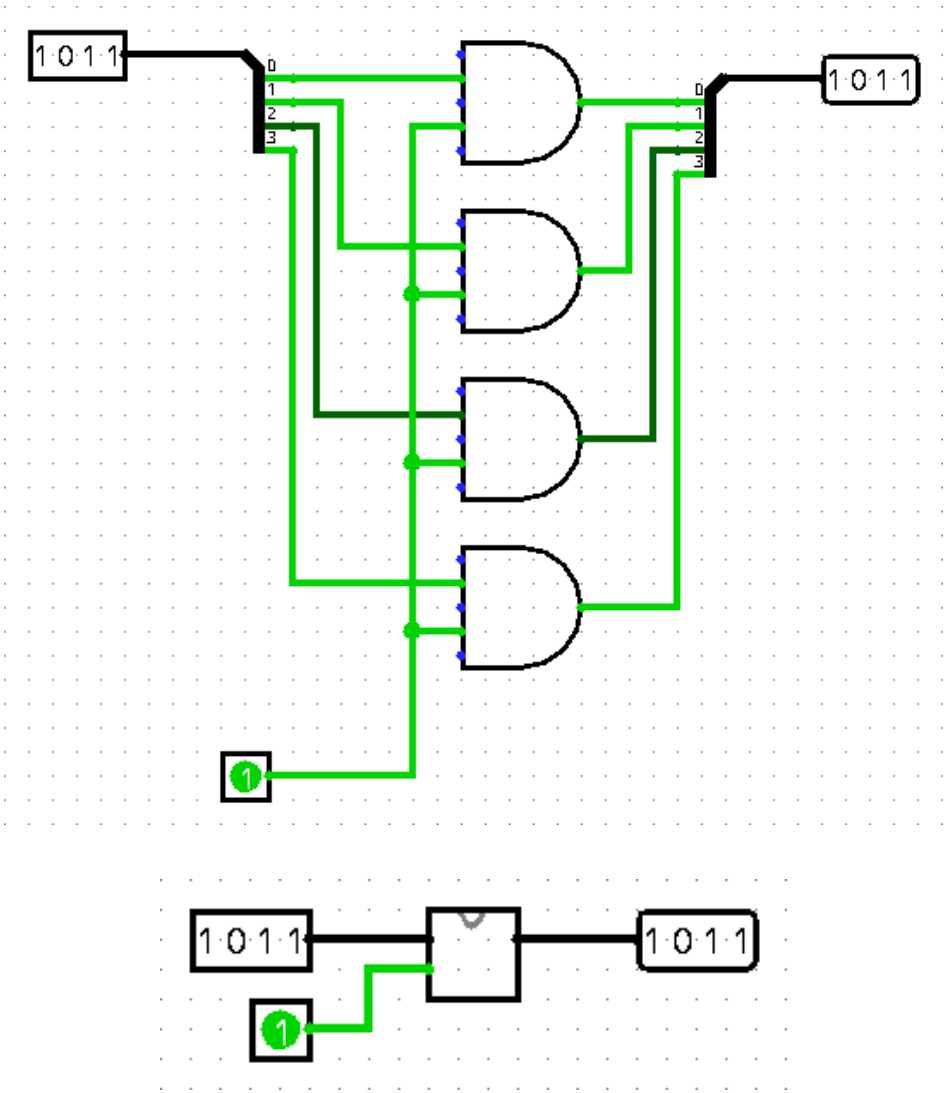


Single digit multiplier



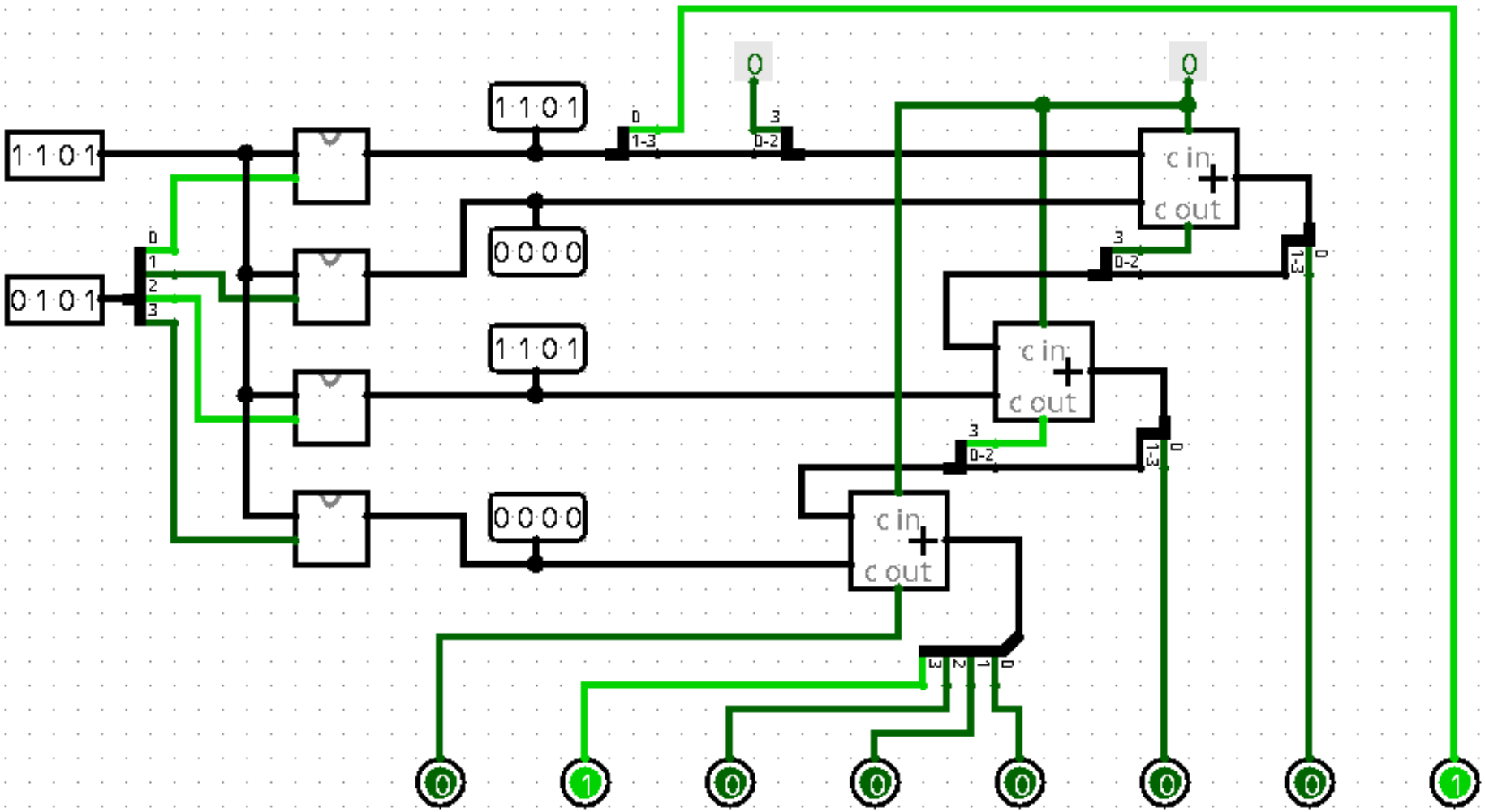


Single digit multiplier



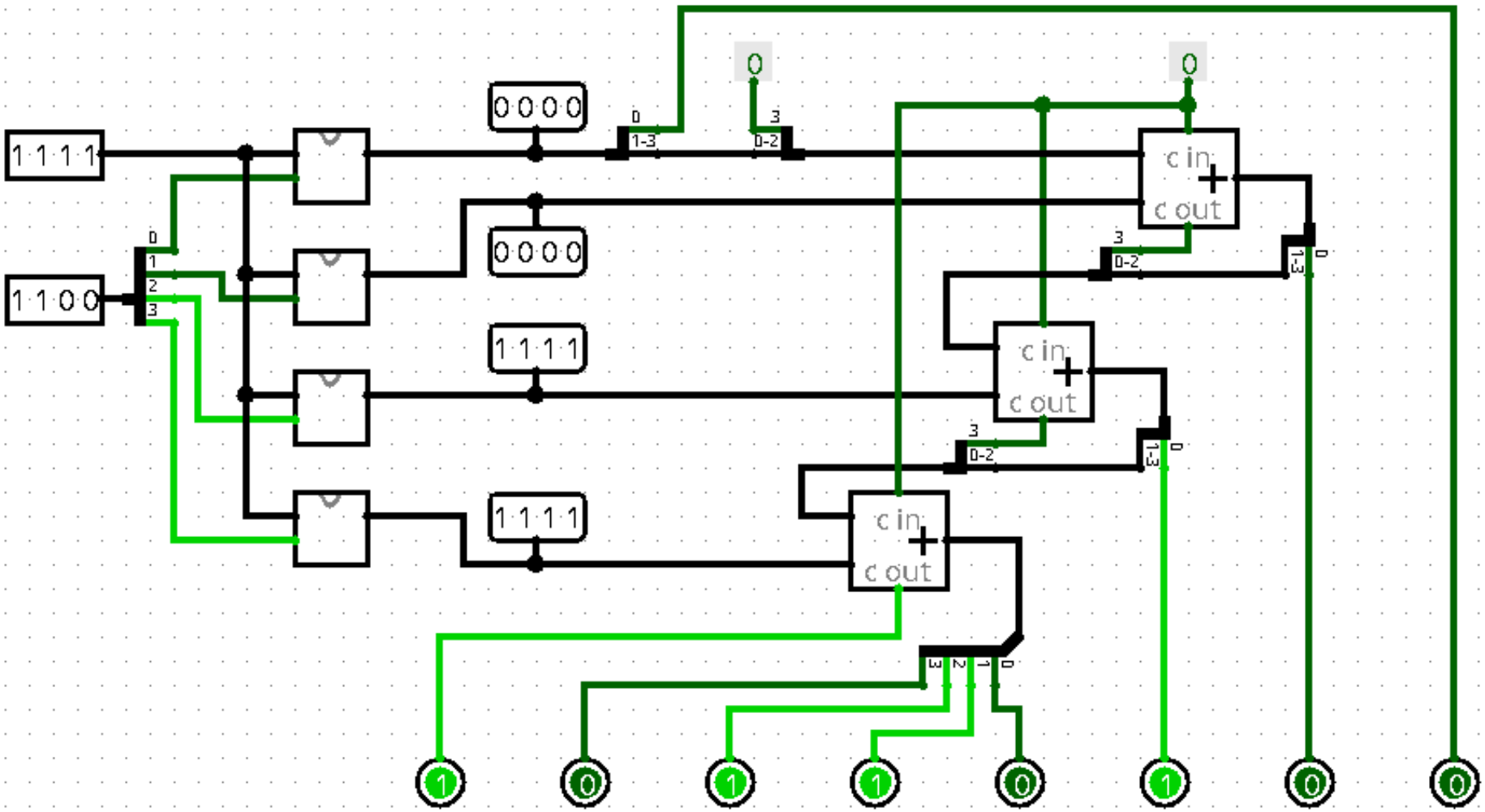
4-bit naïve multiplier

$1101_2 \times 0101_2 = 0100\,0001_2 \quad D_{16} \times 5_{16} = 41_{16} \quad 13_{10} \times 5_{10} = 65_{10}$



4-bit naïve multiplier

$$1111_2 \times 1100_2 = 1011\,0100_2 \quad \mathrm{F}_{16} \times \mathrm{C}_{16} = \mathrm{B4}_{16} \quad 15_{10} \times 12_{10} = 180_{10}$$



Subtraction

Subtracting smaller from larger is easy:

$$1101_2 - 100_2 = 1001_2$$

$$1101_2 - 11_2 = 11\overset{1}{0}1_2 - 11_2 = 1010_2$$

$$10_2 - 11_2 = ???$$

What happens if we add?

$$\begin{array}{cccccc} & & 0 & 0 & 0 & 1 \\ + & & 1 & 1 & 1 & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 \end{array}$$

Negative numbers

Add large numbers so that a carry is generated:

$$\begin{array}{cccccc} & & 0 & 0 & 0 & 1 \\ + & & 1 & 1 & 1 & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 \end{array}$$

i.e.:

$$1111_2 + 1_2 = 0_2$$

but we know that

$$(-1) + 1 = 0$$

Can we say that $1111_2 = -1_2$ in some sense?

Two's complement

Adding 1111 means: $1111_2 = 1\,0000_2 - 1 = 2^4 - 1$

Disregarding the carry digit means subtracting $1\,0000_2 = 2^4$

Thus

$$\begin{aligned} 1 + 1111_2 \text{ (disregarding carry)} &= 1 + (1\,0000_2 - 1) - 1\,0000_2 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Let's see – it works for any number of N bits:

$$a + \underbrace{(2^N - b)}_{\text{2's complement of } b} - 2^N = a - b$$

Take home messages

- Binary is a natural positional number system for digital computations; simple algorithms exist to convert numbers between binary and other number systems, but systems with bases 8 and 16 are the most convenient to represent binary numbers.
- Perl one-liners can be useful for number conversions;
- Not every conversion can be made exactly!
- Not always a two-state signal carries 1 bit of information!
- We can construct digital logic circuits to perform binary arithmetic (addition, subtraction, multiplication, etc.)

References I

Mitašiūnas, Antanas (2016). *Kompiuterių architektūra*. URL: http://www.mif.vu.lt/katedros/cs/Asmen/Kompiuteriu%20architektura.pdf.