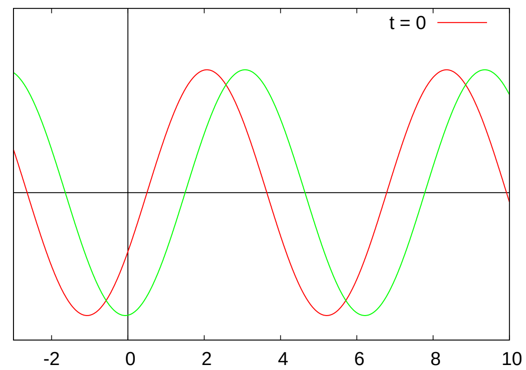
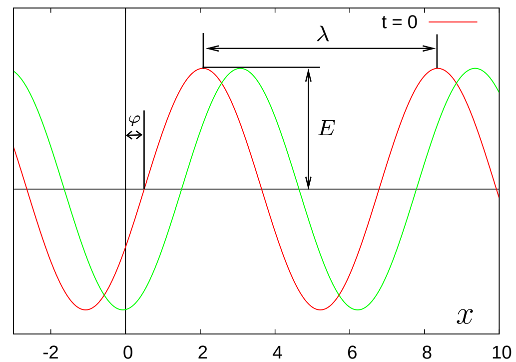
Sin wave



$$y = E\sin(kx - \omega t - \varphi) \tag{1}$$

Sin wave and its parameters



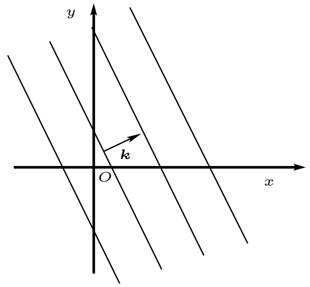
$$y = E\sin(kx - \omega t - \varphi) \tag{2}$$

$$k \underset{def}{=} \frac{2\pi}{\lambda} \tag{3}$$

$$\omega \underset{def}{=} \frac{2\pi}{T} \tag{4}$$

Flat monochromatic wave in space

A flat monochromatic wave in space can be given by the following vector equation:



$$E(\boldsymbol{r}, t) = E_0 \sin(\boldsymbol{kr} - \omega t - \varphi) \tag{5}$$

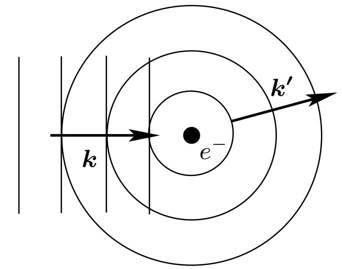
Superposition of two monochromatic waves

Amplitutes of two elecromagnetic waves add up:

$$E = E_1 + E_2 \tag{6}$$

Scattering by an electron

Thomson showed that:



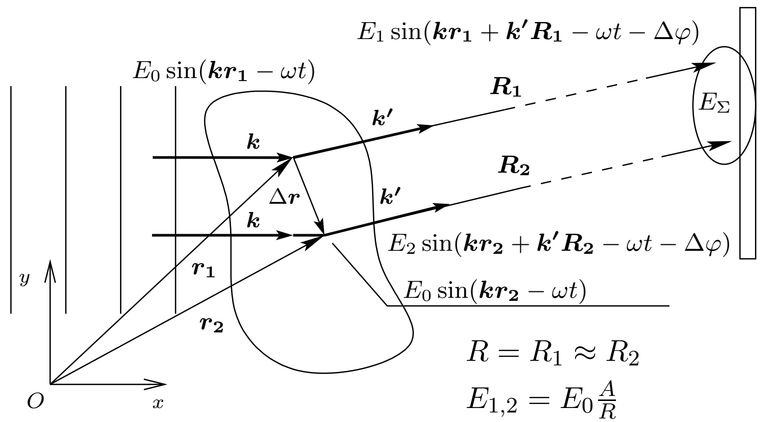
$$E_1(\boldsymbol{R}, t) = E_1 \sin(\boldsymbol{k'}\boldsymbol{R} - \omega t - \Delta\varphi) \quad (7)$$

$$E_1 = E_0 \frac{1}{R}\frac{e^2}{mc^2}\sin\phi = E_0\frac{A}{R} \quad (8)$$

$$m_{p^+} \approx 1800 \cdot m_{e^-}$$

$$I_1 \approx 2\% \cdot I_0$$

Scattering by a specimen



$$E_\Sigma = E_1 \sin(\boldsymbol{kr_1} + \boldsymbol{k'R_1} - \omega t - \Delta\varphi) + E_2 \sin(\boldsymbol{kr_2} + \boldsymbol{k'R_2} - \omega t - \Delta\varphi) \quad (9)$$

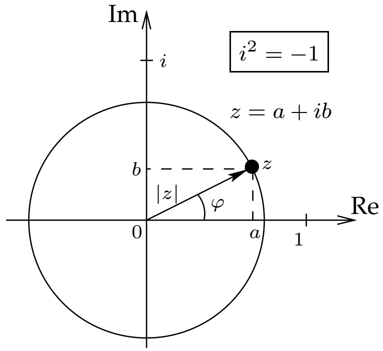
Complex exponents

It is more convenient in physics to represent waves by complex exponents, using Euler's formula:

$$e^{ix} = \cos x + i \sin x \quad (10)$$

We just agree that instead of $\sin$ function, we write a complex exponent and always keep in mind that our real wave is just the imaginary part. Or real. If we care... :)

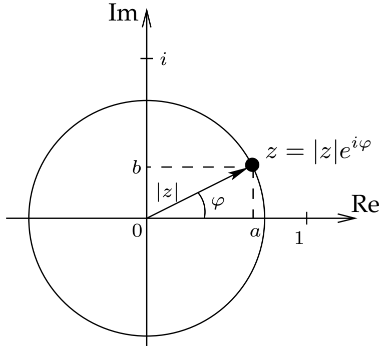
Reminiscences: complex numbers ...



$$z = a + ib \qquad |z| = \sqrt{a^2 + b^2} \qquad \tan\varphi = \frac{b}{a}$$

$$z^* = a - ib \qquad |z|^2 = z \cdot z^*$$

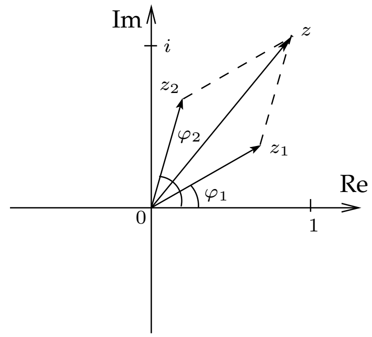
Complex number as exponents



$$|z| = \sqrt{a^2 + b^2} \qquad a = |z|\cos\varphi \qquad b = |z|\sin\varphi$$

$$z = a + ib = |z|\cos\varphi + i|z|\sin\varphi = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$$

Reminiscences: complex numbers as vectors ...



$$z = z_1 + z_2 = a + ib = (a_1 + a_2) + i(b_1 + b_2) \tag{11}$$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \tag{12}$$

Useful properties of exponents

$$e^a e^b = e^{a+b} \tag{13}$$

$$e^{ix} e^{i\varphi} = e^{i(x+\varphi)} \tag{14}$$

Outline of Euler's formula proof...

$$e^{ix} = \cos x + i\sin x$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + \dots$$

$$i\sin x = ix + \frac{1}{3!}(ix)^3 + \frac{1}{5!}(ix)^5 + \dots + \underbrace{(-1)^k i^{-2k-1} i}_{=1} \frac{1}{(2k+1)!}(ix)^{2k+1} + \dots$$

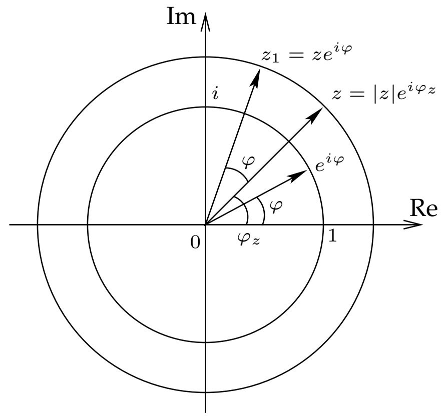
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots + (-1)^k \frac{1}{(2k)!}x^{2k} + \dots$$

$$= 1 + \frac{1}{2!}(ix)^2 + \frac{1}{4!}(ix)^4 + \dots + \underbrace{(-1)^k i^{-2k}}_{=1} \frac{1}{(2k)!}(ix)^{2k} + \dots$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots + \frac{1}{k!}x^k + \dots$$

$$e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \dots + \frac{1}{k!}(ix)^k + \dots$$

Phase shift



$$z_1 = ze^{i\varphi} = |z|e^{i\varphi_z}e^{i\varphi} = |z|e^{i(\varphi_z+\varphi)} \tag{15}$$

Adding two exponents

$$\begin{aligned} e^{ix} + e^{iy} &= e^{ix}(1 + e^{i(y-x)}) \\ &= e^{ix}e^{i\frac{y-x}{2}}(e^{-i\frac{y-x}{2}} + e^{i\frac{y-x}{2}}) \\ &= e^{i2\frac{x}{2}}e^{i\frac{y-x}{2}}(\cos\frac{y-x}{2} - i\sin\frac{y-x}{2} + \cos\frac{y-x}{2} + i\sin\frac{y-x}{2}) \\ &= e^{i\frac{x+y}{2}} \cdot 2\cos\frac{y-x}{2} \end{aligned} \tag{16}$$

Square of the amplitude

let

$$z = |z|e^{i\varphi}\,, \tag{17}$$

then

$$\begin{aligned} z \cdot z^* &= |z|e^{i\varphi} \cdot |z|e^{-i\varphi} \\ &= |z|^2 \underbrace{e^{\overbrace{i(\varphi-\varphi)}^{=0}}}_{=1} \\ &= |z|^2 \end{aligned} \tag{18}$$

Representing waves by exponents

$$E\sin(\boldsymbol{kr} - \omega t) = \mathrm{Im}(Ee^{i(\boldsymbol{kr}-\omega t)}) \tag{19}$$

$$E\sin(\boldsymbol{kr} - \omega t - \varphi) = \mathrm{Im}(Ee^{i(\boldsymbol{kr}-\omega t-\varphi)}) \tag{20}$$

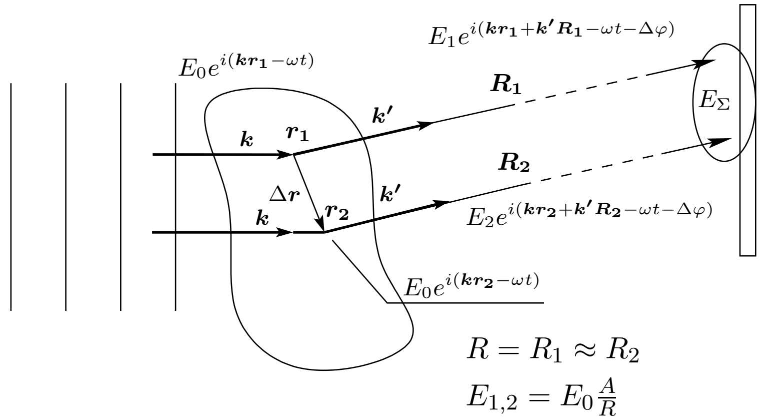
$$= \mathrm{Im}(Ee^{-i\varphi}e^{i(\boldsymbol{kr}-\omega t)}) \tag{21}$$

$$\begin{aligned} &E_1\sin(\boldsymbol{kr} - \omega t - \varphi_1) + \\ &E_2\sin(\boldsymbol{kr} - \omega t - \varphi_2) = \mathrm{Im}(E_1e^{i(\boldsymbol{kr}-\omega t-\varphi_1)} + E_2e^{i(\boldsymbol{kr}-\omega t-\varphi_2)}) \\ &\qquad = \mathrm{Im}((E_1e^{-i\varphi_1} + E_2e^{-i\varphi_2})e^{i(\boldsymbol{kr}-\omega t)}) \end{aligned} \tag{22}$$

(23)

Scattering by a specimen revisited

exponential form:



$$E_\Sigma = E_1 e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1} - \omega t - \Delta\varphi)} + E_2 e^{i(\boldsymbol{k r_2} + \boldsymbol{k' R_2} - \omega t - \Delta\varphi)} \qquad (24)$$

Adding two monochromatic waves

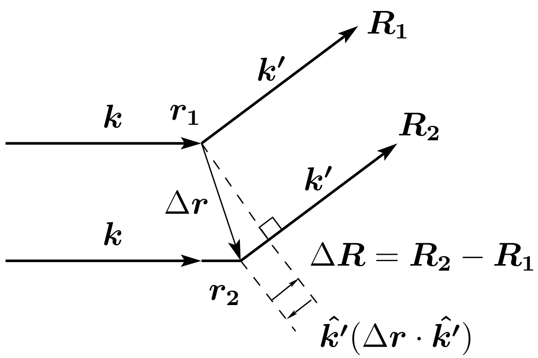
$$E_1(\boldsymbol{R_1}, t) = E_1 e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1} - \omega t - \Delta\varphi)}$$
$$= \underbrace{E_0 \frac{A}{R} e^{-i\Delta\varphi} e^{-i\omega t}}_{\underset{def}{=} E_0'} e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1})}$$
$$= E_0' e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1})} \qquad (25)$$
$$E_2(\boldsymbol{R_2}, t) = E_0' e^{i(\boldsymbol{k r_2} + \boldsymbol{k' R_2})} \qquad (26)$$

$$E_\Sigma = E_0' e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1})} + E_0' e^{i(\boldsymbol{k r_2} + \boldsymbol{k' R_2})}$$
$$= E_0' [e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1})} + e^{i(\boldsymbol{k r_2} + \boldsymbol{k' R_2})}] \qquad (27)$$
$$= E_0' e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1})} [1 + e^{i(\boldsymbol{k r_2} - \boldsymbol{k r_1} + \boldsymbol{k' R_2} - \boldsymbol{k' R_1})}]$$

Adding two monochromatic waves (2)

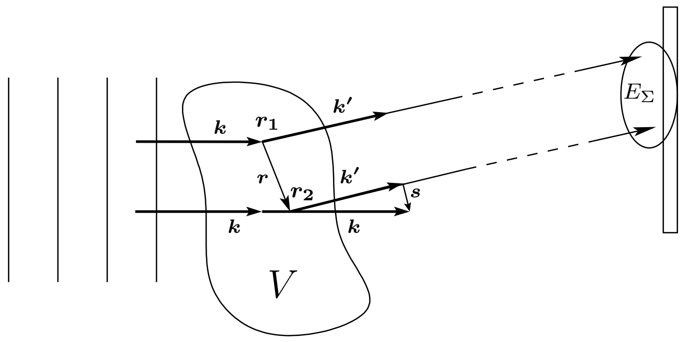
$$E_\Sigma = \underbrace{E_0' e^{i(\boldsymbol{k r_1} + \boldsymbol{k' R_1})}}_{= E_0''} [1 + e^{i(\boldsymbol{k r_2} - \boldsymbol{k r_1} + \boldsymbol{k' R_2} - \boldsymbol{k' R_1})}] \qquad (28)$$
$$= E_0'' [1 + e^{i(\boldsymbol{k} \Delta \boldsymbol{r} + \boldsymbol{k'} \Delta \boldsymbol{R})}]$$

Projection of Δr



$$\hat{\boldsymbol{k}}' = \frac{\boldsymbol{k'}}{k'} \qquad \Delta \boldsymbol{R} = -\hat{\boldsymbol{k}}'(\Delta \boldsymbol{r} \cdot \hat{\boldsymbol{k}}')$$
$$\boldsymbol{k'} \Delta \boldsymbol{R} = -\boldsymbol{k'} \cdot \frac{\boldsymbol{k'}}{k'} (\Delta \boldsymbol{r} \cdot \frac{\boldsymbol{k'}}{k'}) = -\frac{(\boldsymbol{k'} \cdot \boldsymbol{k'})}{k'^2} (\Delta \boldsymbol{r} \cdot \boldsymbol{k'}) = -\boldsymbol{k'} \Delta \boldsymbol{r}$$

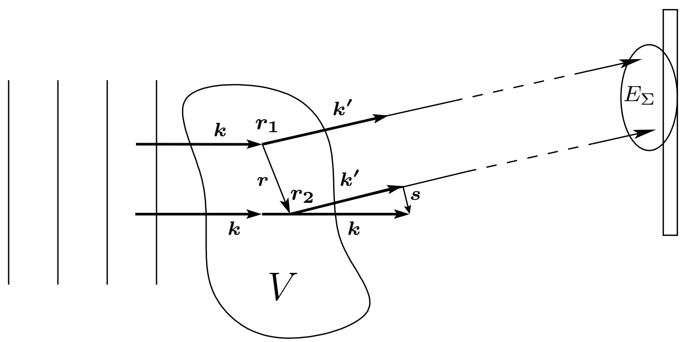
Fourier transform



$$E_\Sigma = \lim_{\substack{N\to\infty \\ \Delta V_k \to 0}} E_0'' \sum_{k=1}^{N} \rho_k \Delta V_k e^{i\boldsymbol{r}_k \boldsymbol{s}} \tag{31}$$

$$= E_0'' \int_V \rho(\boldsymbol{r}) e^{i\boldsymbol{r}\boldsymbol{s}} dV \tag{32}$$

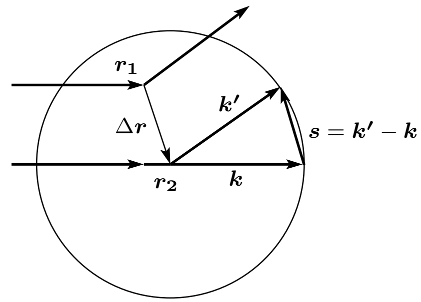
Fourier transform (2)



$$\boldsymbol{S} = \frac{\boldsymbol{s}}{2\pi} \tag{33}$$

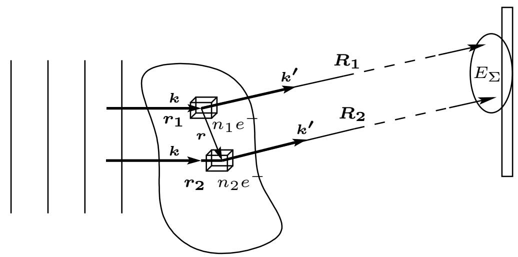
$$E_\Sigma = E_0'' \int_V \rho(\boldsymbol{r}) e^{2\pi i \boldsymbol{r}\boldsymbol{S}} dV \tag{34}$$

Adding two monochromatic waves (3)



$$\begin{aligned} E_\Sigma &= E_0''[1 + e^{i(\boldsymbol{k}\Delta\boldsymbol{r} - \boldsymbol{k'}\Delta\boldsymbol{r})}] \\ &= E_0''[1 + e^{i(\Delta\boldsymbol{r}(\boldsymbol{k}-\boldsymbol{k'}))}] \\ &= E_0''[1 + e^{i\Delta\boldsymbol{r}\boldsymbol{s}}] \\ &= E_0''[1 \cdot e^{i\Delta r_0 s} + e^{i\Delta\boldsymbol{r}\boldsymbol{s}}] \end{aligned} \tag{29}$$

Adding two monochromatic waves (4)



$$\begin{aligned} E_{1+2} &= E_0''[n_1 e^{i r_0 s} + n_2 e^{i\boldsymbol{r}\boldsymbol{s}}] \\ E_\Sigma &= E_0'' \sum_{k=1}^{N} n_k e^{i\boldsymbol{r}_k\boldsymbol{s}} \\ &= E_0'' \sum_{k=1}^{N} \rho_k \Delta V_k e^{i\boldsymbol{r}_k\boldsymbol{s}} \end{aligned} \tag{30}$$

Structure factor

$$E_\Sigma = E_0'' \underbrace{\int_V \rho(\boldsymbol{r}) e^{2\pi i \boldsymbol{r}\boldsymbol{S}} dV}_{F(\boldsymbol{S})} \tag{35}$$

$$\boxed{\mathcal{F}[\rho()](\boldsymbol{S}) = F(\boldsymbol{S}) = \int_V \rho(\boldsymbol{r}) e^{2\pi i \boldsymbol{r}\boldsymbol{S}} dV} \tag{36}$$

Inverse Fourier transform

let

$$F(\boldsymbol{S}) = \int_{-\infty}^{+\infty} \rho(\boldsymbol{r}) e^{2\pi i \boldsymbol{r}\boldsymbol{S}} dV \tag{37}$$

then

$$\rho(\boldsymbol{r}) = \int_{-\infty}^{+\infty} F(\boldsymbol{S}) e^{-2\pi i \boldsymbol{r}\boldsymbol{S}} d\boldsymbol{S} \tag{38}$$

Properties of the Fourier transform

- Linearity

$$\mathcal{F}[\alpha\rho_1 + \beta\rho_2] = \alpha\mathcal{F}[\rho_1] + \beta\mathcal{F}[\rho_2] \tag{39}$$

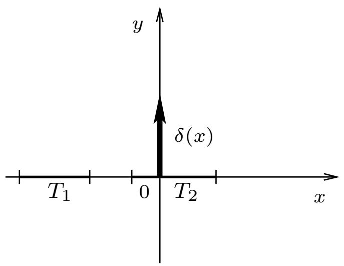
- Shift

$$\mathcal{F}[\rho(\boldsymbol{r} - \Delta\boldsymbol{r})] = e^{2\pi i(\Delta\boldsymbol{r}\boldsymbol{S})}\,\mathcal{F}[\rho(\boldsymbol{r})] \tag{40}$$

- Rotation

$$\boldsymbol{r}' = ||R||\boldsymbol{r}$$
$$\rho'(\boldsymbol{r}) \underset{def}{=} \rho(\boldsymbol{r}') = \rho(||R||\boldsymbol{r})$$
$$\mathcal{F}[\rho(||R||\boldsymbol{r})] = F(||R||\boldsymbol{S}) \tag{41}$$

Dirac delta function



$$\int_{T_1} \delta(x) f(x) dx \underset{def}{=} 0 \tag{42}$$

$$\int_{T_2} \delta(x) f(x) dx \underset{def}{=} f(0) \tag{43}$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x) dx \underset{def}{=} f(0) \tag{44}$$

PROPERTIES OF THE DELTA FUNCTION

$$\delta(-x) = \delta(x) \tag{45}$$

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1 \tag{46}$$

$$\int_{-\infty}^{+\infty} \delta(x - x_0)f(x)dx = f(x_0) \tag{47}$$

$$\mathcal{F}[\delta] = \int_{-\infty}^{+\infty} \delta(x)e^{2\pi ixS}dx = e^0 = 1 \tag{48}$$

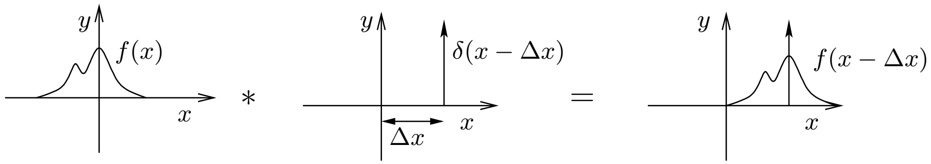
$$\mathcal{F}^{-1}[1] = \int_{-\infty}^{+\infty} e^{-2\pi ixS}dS = \delta(x) \tag{49}$$

$$\mathcal{F}[1] = \int_{-\infty}^{+\infty} e^{2\pi ixS}dx = \delta(S) \tag{50}$$

CONVOLUTION

$$(f * g)(x) \underset{def}{=} \int_{-\infty}^{+\infty} f(v)g(x - v)dv \tag{51}$$

CONVOLUTION WITH DELTA FUNCTION



$$f(x) * \delta(x - \Delta x) = \int_{-\infty}^{+\infty} f(v)\delta(x - v - \Delta x)dv = f(x - \Delta x) \tag{52}$$

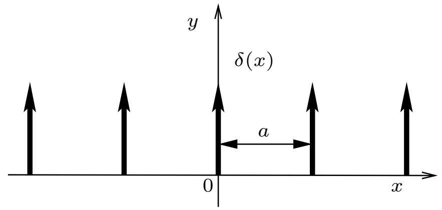
FOURIER TRANSFORM OF THE CONVOLUTION

$$\begin{aligned}
\mathcal{F}[f * g] &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(v)g(x - v)dv \right) e^{2\pi iSx}dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(v)g(x - v)e^{2\pi iS(x-v)}e^{2\pi iSv}dvdx \\
&= \int_{-\infty}^{+\infty} f(v)e^{2\pi iSv}dv \cdot
\end{aligned}$$

$$\int_{-\infty}^{+\infty} g(x - v)e^{2\pi iS(x-v)}d(x - v) \tag{53}$$

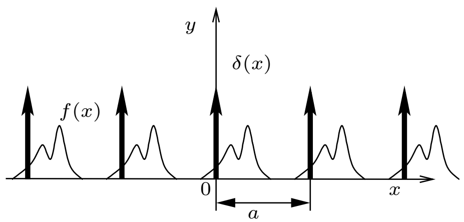
$$= \mathcal{F}[f] \cdot \mathcal{F}[g] \tag{54}$$

Lattice function



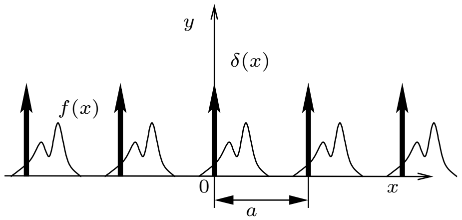
$$L(x) = \sum_{n=-\infty}^{+\infty} \delta(x - an) \tag{55}$$

Crystal function



$$f * L = \sum_{n=-\infty}^{+\infty} f(x - an) \tag{56}$$

Crystal Fourier transform



$$\begin{aligned}
\mathcal{F}[f * L] &= \mathcal{F}[f] \cdot \mathcal{F}[L] && (57)\\
&= \mathcal{F}[\sum_{n=-\infty}^{+\infty} f(x - an)] && (58)\\
&= \sum_{n=-\infty}^{+\infty} \mathcal{F}[f(x - an)] && (59)
\end{aligned}$$

Crystal Fourier transform (2)

$$\begin{aligned}
\mathcal{F}[f * L] &= \mathcal{F}[f] \cdot \mathcal{F}[L] && (60)\\
&= F(S) \cdot \mathcal{F}[L] && (61)\\
&= \mathcal{F}[\sum_{n=-\infty}^{+\infty} f(x - an)] && (62)\\
&= \sum_{n=-\infty}^{+\infty} \mathcal{F}[f(x - an)] && (63)\\
&= \sum_{n=-\infty}^{+\infty} F(S) e^{2\pi i S a n} && (64)\\
&= F(S) \sum_{n=-\infty}^{+\infty} e^{2\pi i S a n} && (65)
\end{aligned}$$

Crystal Fourier transform (3)

$$\sum_{n=-\infty}^{\infty} f(x-an) \underset{def}{=} S(x) \tag{66}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i \cdot n \frac{x}{a}} \tag{67}$$

$$c_n = \frac{1}{a}\int_{-a/2}^{a/2} S(x) e^{2\pi i \cdot n \frac{x}{a}} dx \tag{68}$$

$$= \frac{1}{a}\int_{-a/2}^{a/2} \sum_{n=-\infty}^{\infty} f(x-an) e^{2\pi i \cdot n \frac{x}{a}} dx \tag{69}$$

$$= \frac{1}{a}\sum_{n=-\infty}^{\infty} \int_{-a/2}^{a/2} f(x-an) e^{2\pi i \cdot n \frac{x}{a}} dx \tag{70}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(x) e^{2\pi i \cdot x \frac{n}{a}} dx \tag{71}$$

$$= \frac{1}{a} F(n/a) \tag{72}$$

Crystal Fourier transform (4)

$$c_n = \frac{1}{a} F(n/a) \tag{73}$$

$$\sum_{n=-\infty}^{\infty} f(x-an) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i \cdot n \frac{x}{a}} \tag{74}$$

$$= \frac{1}{a}\sum_{n=-\infty}^{\infty} F(n/a) e^{-2\pi i \cdot n \frac{x}{a}} \tag{75}$$

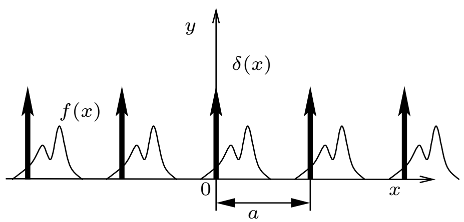
$$\sum_{n=-\infty}^{\infty} \delta(x-an) = \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{-2\pi i \cdot n \frac{x}{a}} \tag{76}$$

Lattice Fourier transform

$$\sum_{n=-\infty}^{\infty} \delta(x-an) = \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{2\pi i \cdot n \frac{x}{a}} \tag{77}$$

$$\sum_{n=-\infty}^{\infty} \delta(x-n/a) = a\sum_{n=-\infty}^{\infty} e^{-2\pi i \cdot nax} \tag{78}$$

Reciprocal lattice, 1D case



$$L(x) = \sum_{n=-\infty}^{+\infty} \delta(x - an) \tag{84}$$

$$\mathcal{F}[f * L] = \mathcal{F}[f] \cdot \mathcal{F}[L] \tag{85}$$

$$= F(S) \cdot \frac{1}{a} \sum_{n=-\infty}^{\infty} \delta(S - n/a) \tag{86}$$

$$\mathcal{F}[L] = \mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(x - an)] \tag{79}$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - an) e^{2\pi i x S} dx \tag{80}$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - an) e^{2\pi i x S} dx \tag{81}$$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi i a n S} \tag{82}$$

$$= \frac{1}{a} \sum_{n=-\infty}^{\infty} \delta(S - n/a) \tag{83}$$

Reciprocal lattice, 3D case

$$L(x) = \sum_{m,n,p=-\infty}^{+\infty} \delta(\boldsymbol{r} - \boldsymbol{a}m - \boldsymbol{b}n - \boldsymbol{c}p) \tag{87}$$

$$L^*(\boldsymbol{S}) = \mathcal{F}[L] = \frac{1}{V} \sum_{h,k,l=-\infty}^{+\infty} \delta(\boldsymbol{S} - h\boldsymbol{a}^* - k\boldsymbol{b}^* - l\boldsymbol{c}^*) \tag{88}$$

$$\boldsymbol{a}^* = \frac{[\boldsymbol{b} \times \boldsymbol{c}]}{V}, \qquad \boldsymbol{b}^* = \frac{[\boldsymbol{c} \times \boldsymbol{a}]}{V}, \qquad \boldsymbol{c}^* = \frac{[\boldsymbol{a} \times \boldsymbol{b}]}{V} \tag{89}$$

$$V = (\boldsymbol{abc}) = (\boldsymbol{a} \cdot [\boldsymbol{b} \times \boldsymbol{c}]) \tag{90}$$

Laue conditions

$$\mathcal{F}[\rho_{cryst}] = \mathcal{F}[\rho * L] = \mathcal{F}[\rho] \cdot \mathcal{F}[L] \tag{91}$$

$$= F(\boldsymbol{S}) \cdot L^\star(\boldsymbol{S}) \tag{92}$$

$$= F(\boldsymbol{S}) \cdot \frac{1}{V} \sum_{h,k,l=-\infty}^{+\infty} \delta(\boldsymbol{S} - h\boldsymbol{a}^* - k\boldsymbol{b}^* - l\boldsymbol{c}^*) \tag{93}$$

We see that the transformant of the crystal is non-vanishing only for a certain values of $\boldsymbol{S}$

$$(\boldsymbol{S} \cdot \boldsymbol{a}) = h, \qquad (\boldsymbol{S} \cdot \boldsymbol{b}) = k, \qquad (\boldsymbol{S} \cdot \boldsymbol{c}) = l \tag{94}$$

$$h, k, l \in \mathbb{Z} \tag{95}$$

Evald construct and Evald sphere