$$F = \sum_{i,j} l_{ij} \sum_{k} \left(u_{ki} u_{kj} - \delta_{ij} \right) = \sum_{lm} l_{lm} \sum_{k} \left(u_{kl} u_{km} - \delta_{lm} \right)$$

$$\frac{\partial F}{\partial u_{ij}} = \sum_{lm} l_{lm} \sum_{k} \frac{\partial}{\partial u_{ij}} \left(u_{kl} \cdot u_{km} \right) = \sum_{lm} l_{lm} \sum_{k} \left(\frac{\partial u_{kl}}{\partial u_{ij}} u_{km} + u_{kl} \frac{\partial u_{km}}{\partial u_{ij}} \right) =$$

$$= \sum_{lm} l_{lm} \sum_{k} \left(\delta_{ik} \delta_{jl} u_{km} + u_{kl} \delta_{ik} \delta_{jm} \right) =$$

$$= \sum_{lm} l_{lm} \left(\delta_{jl} u_{im} + \delta_{jm} u_{il} \right) = \sum_{lm} l_{lm} u_{im} \delta_{jl} + \sum_{lm} l_{lm} u_{il} \delta_{jm} =$$

$$= \sum_{m} l_{jm} u_{im} + \sum_{l} l_{lj} u_{il} = \sum_{m} l_{jm} u_{mi} + \sum_{l} l_{jl} u_{li} = 2 \sum_{m} l_{jm} u_{mi}$$

$$\frac{\partial F}{\partial u_{ij}} = 2 \sum_{k} u_{ik} l_{kj}$$

Jakobi posūkio algoritmas

$$c^2+s^2=1$$

$$S = \begin{bmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{bmatrix} \qquad J = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad S' = J^T S J$$

$$s_{ij} = s_{ji}$$

$$S' = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} s_{ii} & s_{ij} \\ s_{ij} & s_{jj} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} s_{ii}c + s_{ij}s & -s_{ii}s + s_{ij}c \\ s_{ij}c + s_{jj}s & -s_{ij}s + s_{jj}c \end{bmatrix} =$$

$$= \begin{bmatrix} s_{ii}c^2 + s_{ij}sc + s_{ij}sc + s_{jj}s^2 & -s_{ii}sc + s_{ij}c^2 - s_{ij}s^2 + s_{jj}sc \\ -s_{ii}sc - s_{ij}s^2 + s_{ij}c^2 + s_{jj}sc & s_{ii}s^2 - s_{ij}sc - s_{ij}sc + s_{jj}c^2 \end{bmatrix} =$$

$$= \begin{bmatrix} s_{ii}c^2 + 2s_{ij}sc + s_{jj}s^2 & (s_{jj} - s_{ii})sc + s_{ij}(c^2 - s^2) \\ (s_{jj} - s_{ii})sc + s_{ij}(c^2 - s^2) & s_{ii}s^2 - 2s_{ij}sc + s_{jj}c^2 \end{bmatrix}$$

Jakobi posūkio radimas

$$(s_{jj}-s_{ii})sc + s_{ij}(c^2-s^2)=0 \; ; \quad \frac{sc}{c^2-s^2} = -\frac{s_{ij}}{s_{jj}-s_{ii}}$$

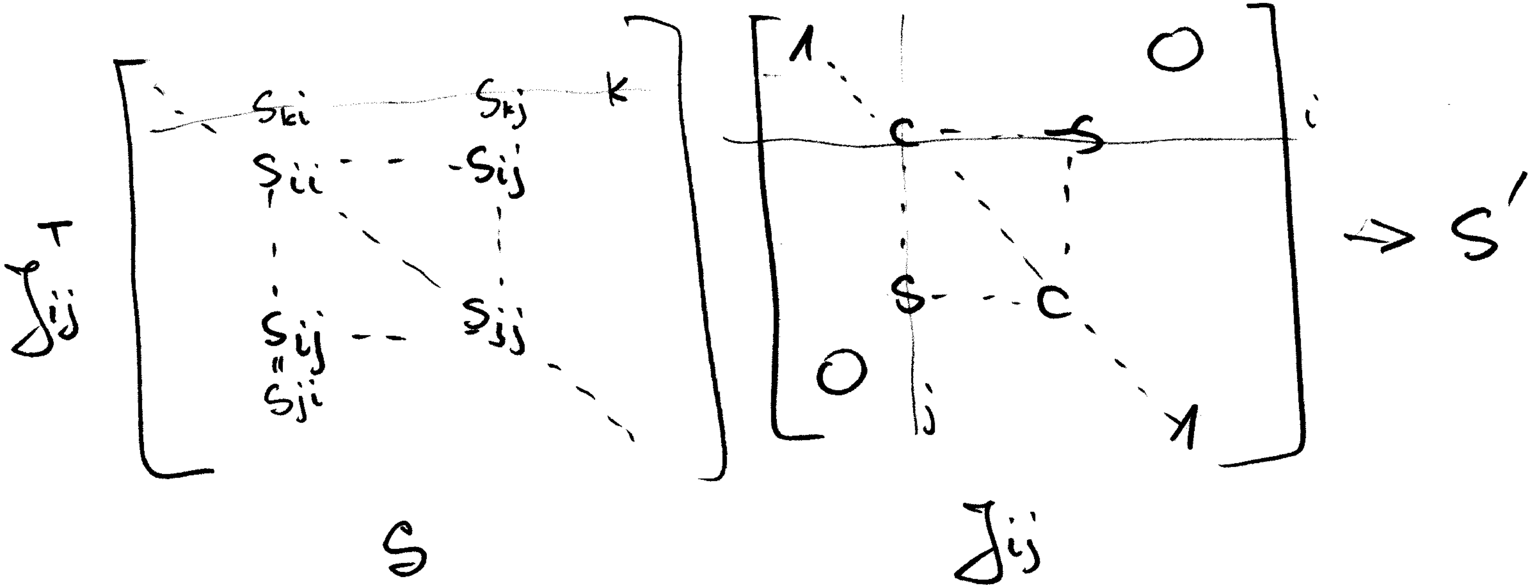
$$sc = \sin\varphi\cos\varphi = \frac{1}{2}\sin 2\varphi$$

$$c^2-s^2 = \cos^2\varphi - \sin^2\varphi = \cos 2\varphi$$

$$\frac{sc}{c^2-s^2} = \frac{\frac{1}{2}\sin 2\varphi}{\cos 2\varphi} = \frac{1}{2}\,\mathrm{tg}\, 2\varphi$$

$$\mathrm{tg}\, 2\varphi = \frac{2s_{ij}}{s_{ii}-s_{jj}}$$

Jakobi iteracija



Daugelio kintamųjų funkcijos ekstremumai ir teigiamos / neigiamos matricos

Funkcijos skleidimas laipsnine (Teiloro) eilute: $x = x_0 + \Delta x$

$$f(x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)\Delta x^2 + o(\Delta x^2)$$

Daugelio kintamųjų atveju:

$$f(\vec{x}) = f(\vec{x}_0) + \operatorname{grad} f \cdot \vec{\Delta x} + \frac{1}{2} \vec{\Delta x}^T \cdot H \cdot \vec{\Delta x}$$

$$= f(x_0) + \left(\vec{\nabla} f, \vec{\Delta x}\right) + \frac{1}{2} \vec{\Delta x}^T \cdot H \cdot \vec{\Delta x}$$

$$H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \\ \vdots & & \\ & & \end{bmatrix}$$ Heso matrica (Hessian)

Jei visos antrosios dalinės išvestinės tolydžios, tada

$$\frac{\partial f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_j \partial x_i},$$ t.y. H yra simetrinė ($H^T = H$)

Kad $f(\vec{x})$ turėtų minimumą taške $\vec{x}_0$, pakanka, kad H būtų teigiama taške $\vec{x}_0$ (t.y. turėtų tik teigiamas tikrines vertes)

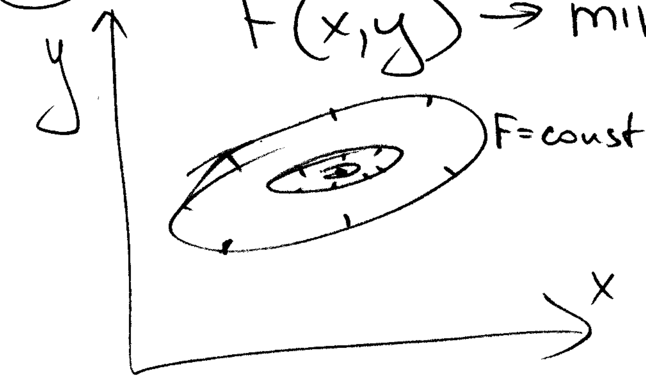
2008.04.21

Daugelio kintamųjų f-jos ekstremumo paieška.

I

$F(x,y) \to min?$

$$\frac{\partial F}{\partial x} = 0; \quad \frac{\partial F}{\partial y} = 0$$ būtina sąlyga



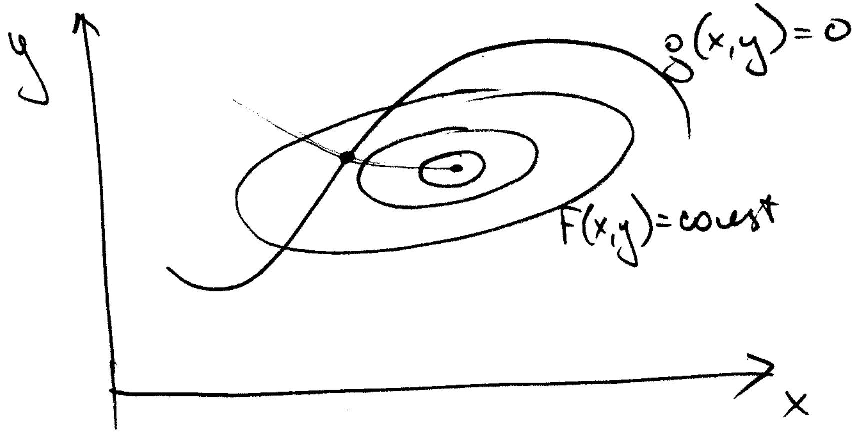
(be apribojimų)

(su apribojimais)

II

$$\begin{cases} F(x,y) \to min \\ g(x,y) = 0 \quad \text{apribojimas} \end{cases}$$

$$\mathcal{F}(x,y,\lambda) = F(x,y) + \lambda g(x,y)$$

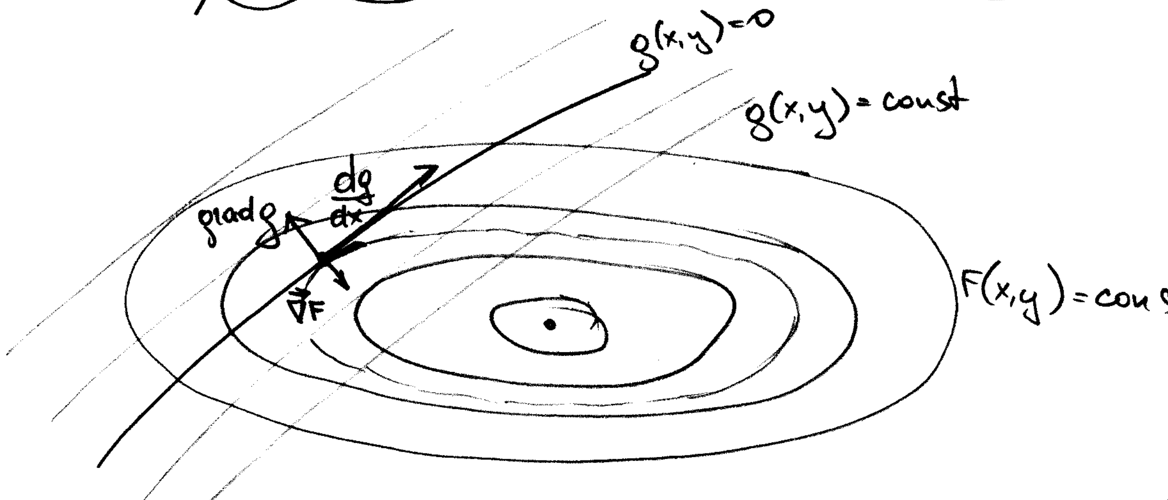


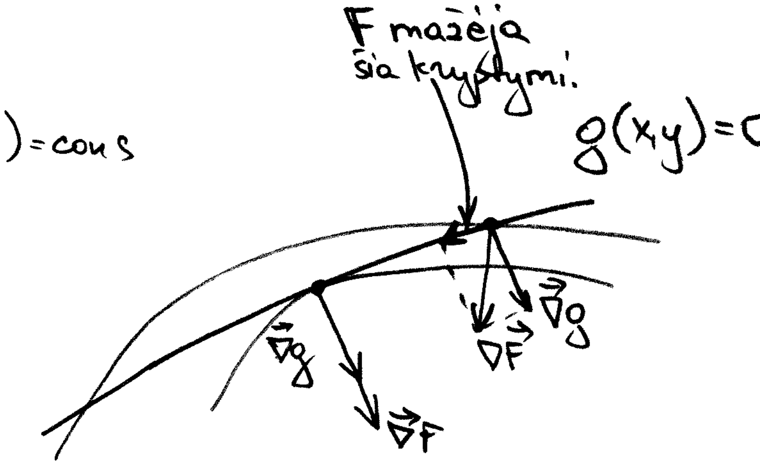
$$\mathcal{F}(x,y,\lambda) \to min$$

!?!

S. GRAŽULIS

Minimumas su apribojimais





$$\vec{\nabla} F = \text{grad}\, F \perp S = \{(x,y) \mid g(x,y)=0\}$$

$$\vec{\nabla} F \parallel \vec{\nabla} g \Rightarrow \vec{\nabla} F = -\lambda \vec{\nabla} g, \quad \lambda \in \mathbb{R}$$

$$\nabla F + \lambda \nabla g = 0$$

$$\frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \lambda\left(\frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y}\right) = 0$$

Lagranžo neapibrėžtų koef. metodas

$$\frac{\partial F}{\partial x}\hat{x}+\frac{\partial F}{\partial y}\hat{y}+\lambda\left(\frac{\partial g}{\partial x}\hat{x}+\frac{\partial g}{\partial y}\hat{y}\right)=0$$

$$\begin{cases}\dfrac{\partial F}{\partial x}+\lambda\dfrac{\partial g}{\partial x}=0\\[2mm] \dfrac{\partial F}{\partial y}+\lambda\dfrac{\partial g}{\partial y}=0\\[2mm] g(x,y)=0\end{cases}$$

$$\mathcal{F}(x,y,\lambda)=F(x,y)+\lambda g(x,y)$$

$$\mathcal{F}(x,y,\lambda)\to\min$$

$$\begin{cases}\dfrac{\partial \mathcal{F}}{\partial x}=0, \quad \dfrac{\partial \mathcal{F}}{\partial y}=0,\\[2mm] \dfrac{\partial \mathcal{F}}{\partial x}=0\end{cases}$$

$$\frac{\partial \mathcal{F}}{\partial x}=\frac{\partial F}{\partial x}+\lambda\frac{\partial g}{\partial x};$$

$$\frac{\partial \mathcal{F}}{\partial y}=\frac{\partial F}{\partial y}+\lambda\frac{\partial g}{\partial y};$$

$$\frac{\partial \mathcal{F}}{\partial \lambda}\equiv g(x,y);$$

$$E = \sum_n \left(U\vec{x}_n - \vec{y}_n\right)^2 = \sum_n \sum_i \left(\sum_k u_{ik}\cdot x_{kn} - y_{in}\right)^2$$

$$\frac{\partial}{\partial u_{ij}} u_{k\ell} = \delta_{ik}\delta_{j\ell}$$

$$\sum_k \delta_{ik} x_k = x_i$$

$$\sum_k \delta_{ik} x_{kn} = x_{in}$$

$$\frac{\partial E}{\partial u_{ij}} = \frac{\partial}{\partial u_{ij}} \sum_n \sum_\ell \left(\sum_k u_{\ell k}\cdot x_{kn} - y_{\ell n}\right)^2 =$$

$$= \sum_n \sum_\ell 2\left\{\left(\sum_k u_{\ell k} x_{kn} - y_{\ell n}\right)\cdot \frac{\partial}{\partial u_{ij}}\left(\sum_k u_{\ell k} x_{kn} - \cancel{y_{\ell n}}\right)\right\} =$$

$$= 2\sum_n \sum_\ell \left\{\left(\sum_k u_{\ell k} x_{kn} - y_{\ell n}\right) \sum_k \frac{\partial u_{\ell k}}{\partial u_{ij}} x_{kn}\right\} =$$

$$= 2\sum_n \sum_\ell \left\{\left(\sum_k u_{\ell k} x_{kn} - y_{\ell n}\right) \sum_k \delta_{i\ell}\,\delta_{jk} x_{kn}\right\} =$$

$$= 2\sum_n \sum_\ell \left\{\left(\overbrace{\sum_k u_{\ell k} x_{kn} - y_{\ell n}}^{z_{\ell n} \to z_{in}}\right) \delta_{i\ell}\, x_{jn}\right\} =$$

$$= 2\sum_n \left(\sum_k u_{ik} x_{kn} - y_{in}\right)\cdot x_{jn} = 2\sum_n \left(\sum_k \left(u_{ik} x_{kn}\cdot x_{jn}\right) - y_{in}\cdot x_{jn}\right) =$$

$$= 2\sum_k u_{ik} \underbrace{\sum_n \left(x_{kn}\cdot x_{jn}\right)}_{\tilde{S}_{kj}} - \underbrace{\sum_n y_{in}\cdot x_{jn}}_{R_{ij}} = 2\left(U\cdot S - R\right) = 0$$

Diferencijuojame $L=[l_{ij}]$

$$F=\frac{1}{2}\sum_{i,j} l_{ij}\Big(\sum_k u_{ki}u_{kj}-\delta_{ij}\Big)=\frac{1}{2}\sum_{m,n} l_{mn}\Big(\sum_k u_{km}u_{kn}-\delta_{mn}\Big)$$

$$\frac{\partial F}{\partial u_{ij}}=\frac{1}{2}\sum_{m,n} l_{mn}\frac{\partial}{\partial u_{ij}}\Big(\sum_k u_{km}u_{kn}-\delta_{mn}\Big)=$$

$$=\frac{1}{2}\sum_{m,n} l_{mn}\sum_k\Big(\frac{\partial u_{km}}{\partial u_{ij}}u_{kn}+u_{km}\frac{\partial u_{kn}}{\partial u_{ij}}\Big)=$$

$$=\frac{1}{2}\sum_{m,n} l_{mn}\sum_k\Big(\delta_{ik}\delta_{jm}u_{kn}+u_{km}\delta_{ik}\delta_{jn}\Big)=$$

$$=\frac{1}{2}\sum_{m,n} l_{mn}\big(\delta_{jm}u_{in}+u_{im}\delta_{jn}\big)=\frac{1}{2}\Big(\sum_{m,n} l_{mn}\delta_{jm}u_{in}+\sum_{m,n} l_{mn}u_{im}\delta_{jn}\Big)$$

$$=\frac{1}{2}\Big(\sum_n l_{jn}u_{in}+\sum_m l_{mj}u_{im}\Big)=\sum_m l_{jm}u_{im}=\sum_m u_{im}l_{mj}=U\cdot L$$

$=$, nes $l_{mj}=l_{jm}$

Diferencijuojame $L = [l_{ij}]$

$$\underline{G = E + F}$$

$$\frac{\partial G}{\partial u_{ij}} = \overbrace{\underbrace{\sum_k u_{ik} \left(\sum_n x_{kn} \cdot x_{jn} - \sum_n y_{in} x_{jn}\right.}_{\frac{\partial E}{\partial u_{ij}}}}^{U \cdot S \quad - \quad R} + \overbrace{\underbrace{\sum_k u_{ik} l_{kj}}_{\frac{\partial F}{\partial u_{ij}}}}^{U \cdot L} =$$

$$= \underbrace{\sum_k u_{ik} \left(\sum_n x_{kn} \cdot x_{jn} + l_{kj} \right)}_{U(S + L)} \underbrace{- \sum y_{in} x_{jn}}_{-R} =$$

$$= U(S+L) - R$$

Ortogonalių matricų (transformacijų) savybės

$$U^T U = U U^T = I$$

Ortogonalios matricos nekeičia vektorių ilgių (atstumų):

$$\vec{a}^2 = |\vec{a}|^2 \underset{def}{=} a^2 \quad ; \quad \vec{a}^2 = (\vec{a}, \vec{a}) = \vec{a}^T \vec{a} = [a_1 \; a_2 \cdots a_n] \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{b} = U\vec{a} \qquad \vec{b}^2 = (\vec{b}, \vec{b}) = \vec{b}^T \vec{b} =$$

$$= (U\vec{a})^T U \cdot \vec{a} = \vec{a}^T \underbrace{U^T U}_{I} \vec{a} =$$

$$= \vec{a}^T \cdot I \cdot \vec{a} = \vec{a}^T \cdot \vec{a} = a^2$$

$$|\vec{a}|^2 = |\vec{b}|^2$$

$$|\vec{a}| = |\vec{b}| \quad \square$$

Matricos tikrinės vertės ir tikriniai vektoriai (eigenvalues, eigenvectors)

$A = [a_{ij}]$ $\updownarrow n$, $\leftrightarrow n$

n-jo laipsnio matrica.

def: $A\vec{x} = \lambda\vec{x}$; $\vec{x} \neq \vec{0}$, $\lambda \in \mathbb{C}$ $(\lambda \in \mathbb{R})$

Matricos tikrinių verčių paieška: charakteringasis daugianaris:

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$ turi sprendinį $\vec{x} \neq 0$

tai įmanoma tik tada, kai

$\det(A - \lambda I) = 0$ (matrica $A - \lambda I$ singuliarinė)

$\det(A - \lambda I)$ yra n-jo laipsnio daugianaris λ atžvilgiu $\Rightarrow$ n-jo laipsnio matrica turi n tikrinių verčių (bendru atveju kompleksinių) (pagrindinė algebros teorema)

Simetrinių matricų savybės

$S^T = S$

1) $A^T A$ visada simetrinė:

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

2) Jei $B = A^T A$, tai B – neneigiama matrica (non-negative definite)!

$$f(\vec{x}) \equiv \vec{x}^T B \vec{x} = \vec{x}^T A^T A \vec{x} = (A\vec{x})^T \cdot (A\vec{x}) = (A\vec{x}, A\vec{x}) = |A\vec{x}|^2 \geq 0$$

3) Neneigiama simetrinė matrica turi realias neneigiamas tikrines vertes:

$B\vec{x} = \lambda \vec{x}$; $\vec{x} \neq \vec{0}$; $\vec{x}^T B \vec{x} \geq 0$ (neneigiama matrica)

$$\vec{x}^T B \vec{x} = \vec{x}^T \lambda \vec{x} = \lambda \vec{x}^T \vec{x} = \lambda x^2 \geq 0$$

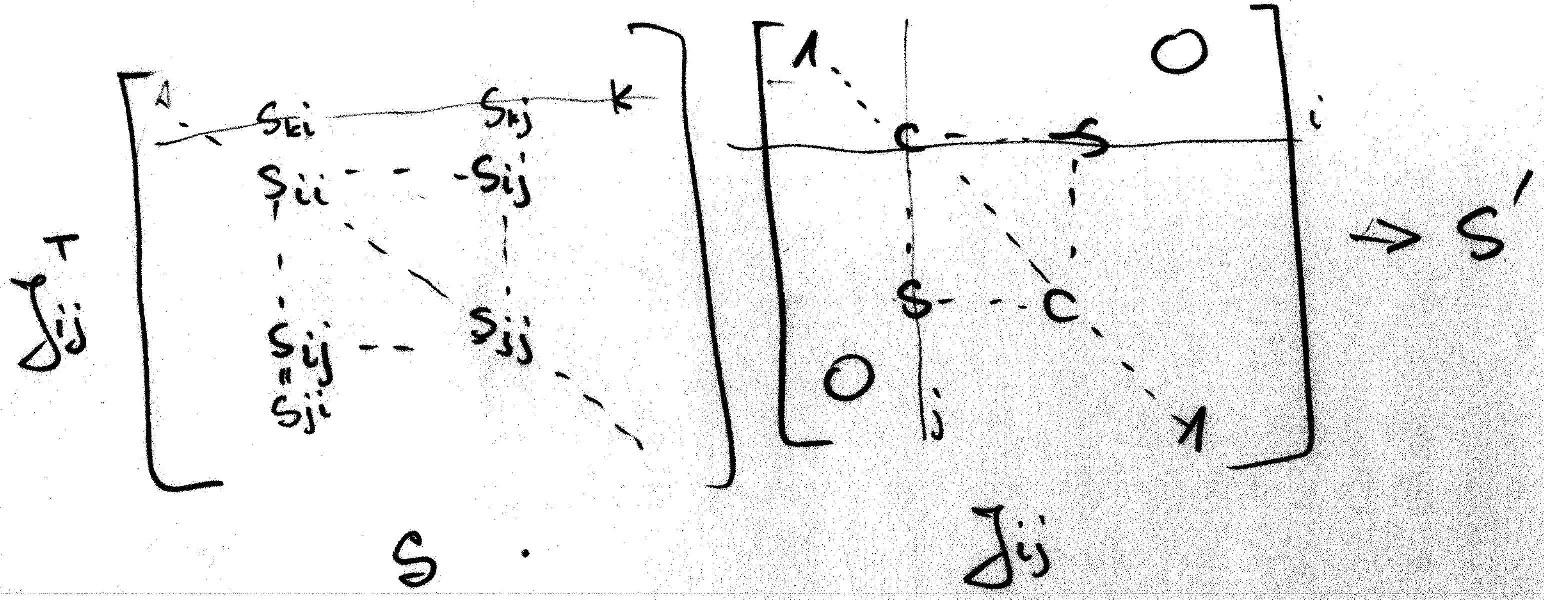
$x^2 \geq 0$ (visada, $\vec{x} \neq \vec{0}$)

$\lambda x^2 \geq 0 \Rightarrow \lambda \geq 0$

Analogiškai: teigiama matrica turi teigiamas tikrines vertes:

$$\left.\begin{array}{l} \vec{x}^T B \vec{x} > 0 \\ B\vec{x} = \lambda x \end{array}\right\} \Rightarrow \lambda > 0$$

Jakobi iteracija



Jakobi posūkio algoritmas

$$c^2+s^2=1$$

$$S=\begin{bmatrix} s_{ii} & s_{ij} \\ s_{ji} & s_{jj} \end{bmatrix} \qquad J=\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad S'=J^T S J$$

$$s_{ij}=s_{ji}$$

$$S'=\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} s_{ii} & s_{ij} \\ s_{ij} & s_{jj} \end{bmatrix}\begin{bmatrix} c & -s \\ s & c \end{bmatrix}=\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} s_{ii}c+s_{ij}s & -s_{ii}s+s_{ij}c \\ s_{ij}c+s_{jj}s & -s_{ij}s+s_{jj}c \end{bmatrix}=$$

$$=\begin{bmatrix} s_{ii}c^2+\underline{s_{ij}sc+s_{ij}sc}+s_{jj}s^2 & -s_{ii}sc+s_{ij}c^2-s_{ij}s^2+s_{jj}sc \\ -s_{ii}sc-\underline{s_{ij}s^2+s_{ij}c^2}+s_{jj}sc & s_{ii}s^2-\underline{s_{ij}sc}-\underline{s_{ij}sc}+s_{jj}c^2 \end{bmatrix}=$$

$$=\begin{bmatrix} s_{ii}c^2+2s_{ij}sc+s_{jj}s^2 & (s_{jj}-s_{ii})sc+s_{ij}(c^2-s^2) \\ (s_{jj}-s_{ii})sc+s_{ij}(c^2-s^2) & s_{ii}s^2-2s_{ij}sc+s_{jj}c^2 \end{bmatrix}$$

Jakobi posūkio radimas

$$(s_{jj}-s_{ii})sc + s_{ij}(c^2-s^2)=0 \; ; \quad \frac{sc}{c^2-s^2} = -\frac{s_{ij}}{s_{jj}-s_{ii}}$$

$$sc = \sin\varphi\cos\varphi = \frac{1}{2}\sin 2\varphi$$

$$c^2-s^2 = \cos^2\varphi - \sin^2\varphi = \cos 2\varphi$$

$$\frac{sc}{c^2-s^2} = \frac{\frac{1}{2}\sin 2\varphi}{\cos 2\varphi} = \frac{1}{2}\,\mathrm{tg}\,2\varphi$$

$$\mathrm{tg}\,2\varphi = \frac{2s_{ij}}{s_{ii}-s_{jj}}$$

Simetrinės matricos tikrinės vertės (matrica su realiais koef.)

$$A\vec{x} = \lambda\vec{x}, \qquad \vec{x} \neq \vec{0}$$

$$A^T = A$$

$$\vec{x}^T A = \lambda \vec{x}^T \quad \left(\text{nes } (A\vec{x})^T = \vec{x}^T A^T = \vec{x}^T A\right)$$

$$\overline{\vec{x}^T A} = \overline{\vec{x}^T} A = \bar{\lambda}\, \overline{\vec{x}^T}$$

$$\overline{\vec{x}^T}\, A\vec{x} = \left(\overline{\vec{x}^T A}\right)\vec{x} = \bar{\lambda}\, \overline{\vec{x}^T}\vec{x}$$

kita vertus,

$$\overline{\vec{x}^T} \underbrace{A\vec{x}}_{\lambda\vec{x}} = \overline{\vec{x}^T} \lambda\vec{x} = \lambda \overline{\vec{x}^T}\vec{x}$$

taigi $\lambda \overline{\vec{x}^T}\vec{x} = \bar{\lambda}\, \overline{\vec{x}^T}\vec{x}$; bet $\overline{\vec{x}^T} \cdot \vec{x} = \langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$

$\mathbb{C}$ aibėje

$$\lambda |\vec{x}|^2 = \bar{\lambda} |\vec{x}|^2; \quad |\vec{x}|^2 \neq 0 \ (\text{nes } \vec{x} \neq \vec{0})$$

$$\lambda = \bar{\lambda} \quad \square$$

Kodėl simetrinės matricos tikriniai vektoriai ortogonalūs?

$i \neq j$

$\lambda_i \neq \lambda_j$

$$(A\vec{x})^T \vec{x} = \cancel{\&x}$$
$$= \vec{x}^T \cdot A^T \vec{x}$$

$$A\vec{x}_i = \lambda_i x_i$$

$$\lambda_i(\vec{x}_i, \vec{x}_j) = (\lambda_i \vec{x}_i, \vec{x}_j) =$$
$$= (A\vec{x}_i, \vec{x}_j) = (\vec{x}_i, A^T \vec{x}_j) =$$
$$= (\vec{x}_i, A\vec{x}_j) = \lambda_j(\vec{x}_i, \vec{x}_j)$$

$A = A^T$!

$$\lambda_i \neq \lambda_j \Rightarrow (x_i, x_j) = 0$$

Simetrinės matricos išskaidymas

$$A = A^T$$

$$A = QDQ^T \qquad QQ^T = I$$

$$D = Q^T A Q$$

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A \qquad \text{OK}$$

Kodėl simetrinės matricos tikrinės vertės realios?

$$\det(A - \lambda I) = 0$$

$$A\vec{x} = \lambda\vec{x}, \quad \vec{x} \neq \vec{0}$$

$$\cancel{\overline{\vec{x}^T A^T = \vec{x}^T A = \lambda \vec{x}^T}}$$

$$\vec{x}^T A = \lambda \vec{x}^T$$

$$\overline{\vec{x}}^T A = \overline{\lambda}\, \overline{\vec{x}}^T$$

$$\overline{\vec{x}}^T \underbrace{A\vec{x}}_{\lambda\vec{x}} = \overline{\lambda}\, \overline{\vec{x}}^T \vec{x}$$

$$\overline{\vec{x}}^T \lambda\vec{x} = \overline{\lambda}\, \overline{\vec{x}}^T \vec{x}$$

$$\lambda(\overline{\vec{x}}^T, \vec{x}) = \overline{\lambda}\underbrace{(\overline{\vec{x}}^T, \vec{x})}_{\in \mathbb{R},\ > 0}$$

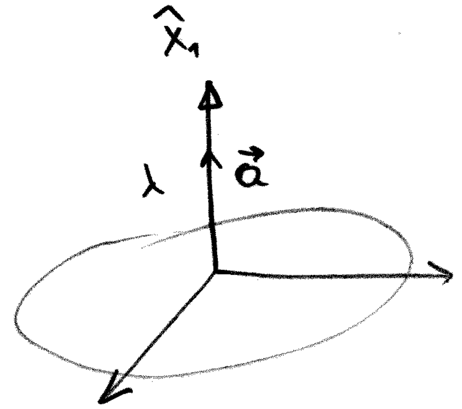
$$\lambda = \overline{\lambda}$$

$$\lambda \in \mathbb{R}$$

$$AQ = QD$$

$$AQ_n = \lambda_n Q_w$$

Q_n is n-th column.



$$\begin{pmatrix} \lambda & 0 \\ 0 & \boxed{B_1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\vec{x}^T \cdot B \cdot \vec{x} = \underbrace{\lambda x_1^2}_{>0} + \vec{x}'^T B_1 \vec{x}' > 0$$

$$\Rightarrow \vec{x}'^T B_1 \vec{x}' > 0$$

$A = A^T \Rightarrow \lambda_i$ yra N reikšmių

visi tikriniai vekt. ortog.

A teigiama $\Rightarrow \lambda_i > 0$

$A^T = A$

$\det(A - \lambda I) = 0$

$Ax = \lambda x$

$\cancel{AAx = \lambda Ax = \lambda^2 x}$

$A^T A = A^2$ simetrinė teigiama matrica

$$U^T U = U U^T = I$$

$\vec{a}^2 = a^2$ $\qquad$ $\vec{a}^2 = (\vec{a}, \vec{a}) = \vec{a}^T \vec{a}$

$\vec{b} = U\vec{a}$: $\qquad$ $\vec{b}^2 = (\vec{b}, \vec{b}) = \vec{b}^T \vec{b} =$

$$= (U\vec{a})^T \cdot U \cdot \vec{a} = \vec{a}^T U^T U \vec{a} = \vec{a}^T \cdot I \cdot \vec{a} =$$

$$= \vec{a}^T \vec{a} = a^2 \Rightarrow$$ ortogonali transformacija nekeičia atstumų

$$A^T = A$$

$$f(\vec{x}, \vec{y}) = \vec{x}^T \cdot A \cdot \vec{y}$$

$$f_A(\vec{x}) = \vec{x}^T A \vec{x}$$

$$\vec{y} = \alpha \vec{a}_1 + \beta \vec{a}_2$$

$$\vec{A} = \lambda_1 \alpha \vec{a}_1 + \beta \lambda_2 \vec{a}_2$$

$$\vec{x}^T A \vec{x}$$

$$f_B(\vec{x}) = \vec{x}^T A^T A \vec{x} = (A\vec{x})^T (A\vec{x}) = (A\vec{x})^2 \geq 0$$

$B = A^T A$ $\qquad$ $B^T = (A^T A)^T = A^T (A^T)^T = A^T A = B$

$$B = A^2$$

B – positive definite $\qquad$ A is any matrix

$B\vec{x} = \lambda \vec{x}$ $\qquad$ $\vec{x} \neq \vec{0}$ $\qquad$ $\Rightarrow$ Positive definite matrix has (real) positive eigenvalues

$$\vec{x} B \vec{x} = \vec{x} \lambda \vec{x} = \lambda x^2 > 0$$

$$x^2 > 0 \Rightarrow \lambda > 0$$

Simetrinės matricos išskaidymas

$$A = A^T$$

$$A = QDQ^T \qquad QQ^T = I$$

$$D = Q^T A Q$$

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A \qquad \text{OK}$$

Kodėl simetrinės matricos tikrinės vertės realios?

$$\det(A - \lambda I) = 0$$

$$A\vec{x} = \lambda\vec{x}, \quad \vec{x} \neq \vec{0}$$

$$\vec{x}^T A^T = \vec{x}^T A = \lambda \vec{x}^T$$

$$\vec{x}^T A = \lambda \vec{x}^T$$

$$\overline{\vec{x}}^T A = \overline{\lambda}\, \overline{\vec{x}}^T$$

$$\overline{\vec{x}}^T \underbrace{A\vec{x}}_{\lambda\vec{x}} = \overline{\lambda}\, \overline{\vec{x}}^T \vec{x}$$

$$\overline{\vec{x}}^T \lambda\vec{x} = \overline{\lambda}\, \overline{\vec{x}}^T \vec{x}$$

$$\lambda \cancel{(\overline{\vec{x}}^T, \vec{x})} = \overline{\lambda} \underbrace{\cancel{(\overline{\vec{x}}^T, \vec{x})}}_{\in \mathbb{R},\, >0}$$

$$\lambda = \overline{\lambda}$$

$$\lambda \in \mathbb{R}$$

$$AQ = QD$$

$$AQ_n = \lambda_n Q_w$$

Q_n is n-th column.

Kodėl simetrinės matricos tikriniai vektoriai ortogonalūs?

$i \neq j$

$\lambda_i \neq \lambda_j$

$(A\vec{x})^T \vec{x} = \cancel{\&x}$

$= \vec{x}^T \cdot A^T \vec{x}$

$A\vec{x}_i = \lambda_i x_i$

$\lambda_i(\vec{x}_i, \vec{x}_j) = (\lambda_i \vec{x}_i, \vec{x}_j) =$

$= (A\vec{x}_i, \vec{x}_j) = (\vec{x}_i, A^T\vec{x}_j) =$

$= (\vec{x}_i, A\vec{x}_j) = \lambda_j(\vec{x}_i, \vec{x}_j)$

$A = A^T$!

$\lambda_i \neq \lambda_j \Rightarrow (x_i, x_j) = 0$