Vector algebra facts sheet.

Notation

1. Vectors will be denoted by bold letters:

$$\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{v}$$

When hand-written, vectors will be denoted with small arrows above the letters:

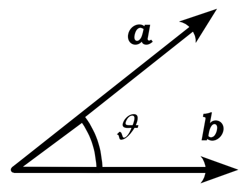
$$\vec{a},\ \vec{b},\ \vec{c}$$

2. Normal typeface letter will indicate a length of a vector:

$$a=|\boldsymbol{a}|$$

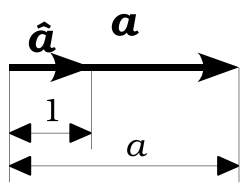
3. An angle between two vectors:

$$\vartheta=\widehat{\boldsymbol{a}\boldsymbol{b}}$$



4. A vector with a "hat" is a unit-length vector pointing in the direction of the original vector:

$$\hat{\boldsymbol{v}}\|\boldsymbol{v},\ |\hat{\boldsymbol{v}}|\equiv 1,\ \boldsymbol{v}\cdot\hat{\boldsymbol{v}}\geqslant 0$$



Vector space

1. A set of all vectors forms a vector space *V*; vectors can be added, for any vector there exists an opposite vector, and there exists a zero vector:

$$\forall\,\boldsymbol{a},\boldsymbol{b}\in V:\ \boldsymbol{c}=\boldsymbol{a}+\boldsymbol{b}\in V$$
$$\exists\boldsymbol{0}\in V:\ \forall\,\boldsymbol{a}+\boldsymbol{0}=\boldsymbol{0}+\boldsymbol{a}=V$$
$$\forall\,\boldsymbol{a}\in V:\ \exists(-\boldsymbol{a})\in V:$$
$$\boldsymbol{a}+(-\boldsymbol{a})=(-\boldsymbol{a})+\boldsymbol{a}=\boldsymbol{0}$$

2. Vector addition is commutative:

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}\ \ (\forall\,\boldsymbol{a},\boldsymbol{b}\in V)$$

3. Vectors from the vector space multiplied by numbers:

$$\forall\,\lambda\in\mathbb{C}:\ \forall\,\boldsymbol{v}\in V:\ \exists(\lambda\boldsymbol{v}):\ (\lambda\,\boldsymbol{v})\in V$$

4. The following vector algebra rules hold (λ and μ are numbers):

$$\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\,\boldsymbol{a}+\lambda\,\boldsymbol{b}$$
$$(\lambda+\mu)\,\boldsymbol{a}=\lambda\,\boldsymbol{a}+\mu\,\boldsymbol{a}$$
$$0\cdot\boldsymbol{a}=\boldsymbol{0}\ \ \text{(zero vector)}$$

Basis

1. Three non-coplanar vectors form a basis:

$$\hat{\boldsymbol{x}}=\boldsymbol{e}_{\boldsymbol{1}},\hat{\boldsymbol{y}}=\boldsymbol{e}_{\boldsymbol{2}},\hat{\boldsymbol{z}}=\boldsymbol{e}_{\boldsymbol{3}}:$$
$$\forall\,i\neq j:\ \neg(\boldsymbol{e}_{\boldsymbol{i}}\|\boldsymbol{e}_{\boldsymbol{j}})$$

2. Any vector can be expressed via the base vectors:

$$\boldsymbol{v}=v_x\,\hat{\boldsymbol{x}}+v_y\,\hat{\boldsymbol{y}}+v_z\,\hat{\boldsymbol{z}}$$

3. Vector can be defined through its components:

$$\boldsymbol{v}=(v_x,v_y,v_z)$$

5. Vector length can be expressed through its components (in an orthonormal basis):

$$v=|\boldsymbol{v}|=\sqrt{v_x^2+v_y^2+v_z^2}$$

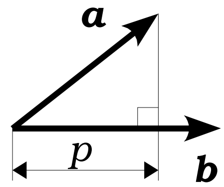
Scalar and vector products

1. Scalar product is a scalar quantity:

$$\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\,\boldsymbol{b}=a\,b\cos\widehat{\boldsymbol{a}\boldsymbol{b}}$$

2. Projection of vector $\boldsymbol{a}$ into direction of vector $\boldsymbol{b}$ can be expressed via the scalar product:

$$p=\frac{\boldsymbol{a}\,\boldsymbol{b}}{b}=\boldsymbol{a}\,\hat{\boldsymbol{b}}=a\cos\widehat{\boldsymbol{a}\,\hat{\boldsymbol{b}}}$$



3. Scalar product is commutative:

$$\boldsymbol{a}\,\boldsymbol{b}=\boldsymbol{b}\,\boldsymbol{a}$$

4. Other scalar product properties:

$$(\boldsymbol{a}+\boldsymbol{b})\,\boldsymbol{c}=\boldsymbol{a}\,\boldsymbol{c}+\boldsymbol{b}\,\boldsymbol{c}$$
$$\lambda(\boldsymbol{a}\cdot\boldsymbol{b})=(\lambda\,\boldsymbol{a}\cdot\boldsymbol{c})$$
$$(\boldsymbol{a}\cdot\boldsymbol{a})=a^2$$

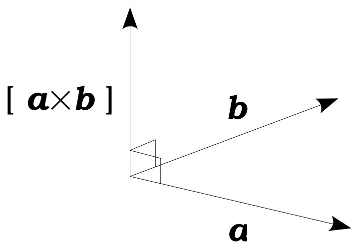
5. Scalar product in orthonormal coordinates:

$$\boldsymbol{a}\,\boldsymbol{b}=a_x b_x+a_y b_y+a_z b_z$$

Vector product

1. Vector product is a vector perpendicular to both multiplicands:

$$\|[\boldsymbol{a}\times\boldsymbol{b}]\|=a\,b\,\sin\widehat{\boldsymbol{a}\,\boldsymbol{b}}$$
$$[\boldsymbol{a}\times\boldsymbol{b}]\perp\boldsymbol{a},\;[\boldsymbol{a}\times\boldsymbol{b}]\perp\boldsymbol{b}$$



2. Vector product can be calculated from the coordinates using the following determinant:

$$[\boldsymbol{a}\times\boldsymbol{b}]=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

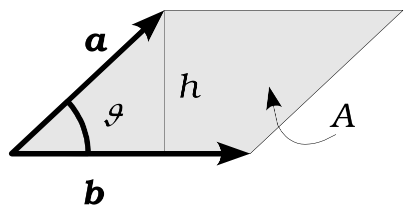
3. Vectors **a**, **b**, and **c** form a right-handed system.

4. Vector product is anticommutative:

$$[\boldsymbol{a}\times\boldsymbol{b}]=-[\boldsymbol{b}\times\boldsymbol{a}]$$

5. Vector product gives an area of parallelepiped spanned by vectors **a** and **b**:

$$A=h\,b=a\,b\,\sin\widehat{\boldsymbol{a}\,\boldsymbol{b}}$$
$$h=a\sin\widehat{\boldsymbol{a}\,\boldsymbol{b}}=a\,\sin\vartheta$$



Triple product

1. Triple product of three vectors is a defined via scalar and vector products:

$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\equiv(\boldsymbol{a}\cdot[\boldsymbol{c}\times\boldsymbol{d}])$$

2. Triple product can be “rotated”:

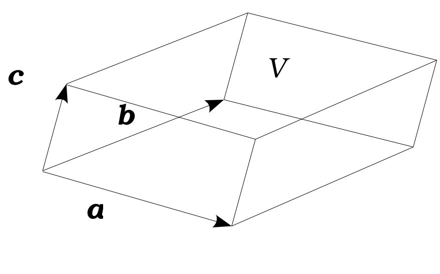
$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a})=(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})$$

3. Triple product changes its sign when any two multiplicands are transposed:

$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=-(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c})=-(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b})$$

4. Triple product gives a volume of a prism spanned by the three vectors:

$$V=(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$$



Double vector product

1. The following formula holds:

   $[\boldsymbol{a} \times [\boldsymbol{b} \times \boldsymbol{c}]] = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$